Favorable-Propagation-Exploited Variational Inference For Massive MIMO Detection

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***Abstract*—In this paper, we investigate the massive MIMO detection under the framework of mean-field variational inference (VI), which leads to a better detection trade-off between performance and complexity. First of all, by fully taking advantages of the favorable propagation characteristic of massive MIMO, the favorable-propagation-exploited variational inference (FPE-VI) algorithm is proposed for the low-complexity detection. Secondly, with respect to the system with K transmitting and N receiving antennas, the linear version of the FPE-VI detection is studied in detail, where its convergence is ensured when $N/K > 1/(\sqrt{2}-1)^2$. Thirdly, by examining the evidence lower bound (ELBO) of the proposed FPE-VI, further optimization via the application of discrete Gaussian distribution is presented for extra performance gain. Finally, all the related theoretical analysis and the improved performance-complexity trade-off of FPE-VI are demonstrated by numerical results.**

***Index Terms*—Massive MIMO detection, approximate inference, variational inference, favorable propagation.**

I. Introduction

With the rapid evolution of massive multiple-input multiple-output (MIMO) system, the exploded scale of the antennas poses a tough challenge on its uplink signal detection, so that developing the methods with low complexity but satisfactory performance attracts the increasing attentions [1]. For this reason, the machine learning technology [2] has brought unprecedented boosting in massive MIMO detection, which is equivalent to solving an inference problem on the posterior distribution of the transmission signal.

Specifically, sum-product is a classic exact inference algorithm derived from the message passing method. Employing it on the loop-free graph leads to the belief propagation, and there are numerous massive MIMO detection schemes based on them [3]–[6]. Besides, the deterministic approximate inference scheme chiefly relies on the analytical approximation to the posterior distribution by factorizing it in a particular way. For instance, expectation propagation (EP) algorithm manages to find the satisfactory distribution alternatively along the individual factors under exponential-family assumption [7].

In principle, EP is a method based on the minimization of the reverse-formed Kullback-Leibler (KL) divergence, i.e., $\mathrm{KL}(p||q)$, where p is the true posterior and q is the approximated one. As an alternative form of the deterministic approximate inference, the variational inference (VI), however, aims to minimize $\mathrm{KL}(q||p)$, which introduces different properties into the approximation. Most importantly, unlike EP, the convergence of VI is guaranteed with a monotonically increasing evidence lower bound (ELBO) [2]. Nevertheless, the VI-based massive MIMO detection still makes little progress. [8] provides a VI framework for multiuser detection, which tries to interpret the traditional zero forcing (ZF), minimum mean square error (MMSE) and successive interference cancellation (SIC) solutions in the way of VI. However, to our knowledge, there is no any particular detection method derived from VI framework.

In this paper, by developing the detection method from VI perspective, the favorable-propagation-exploited variational inference (FPE-VI) algorithm is proposed to achieve a better detection trade-off for massive MIMO systems with K transmitting and N receiving antennas. Typically, based on the favorable-propagation characteristic, the mean-field variational inference method is introduced for a low-complexity detection. Meanwhile, the mean and variance of its linear version are proved to be convergent when the antenna ratio N/K is greater than $1/(\sqrt{2}-1)^2$. Moreover, the initial distribution of the transmitted signal is optimized for an improved detection performance, where the discrete Gaussian probability is employed to strengthen the convergence of VI by increasing ELBO.

II. System Model

For notational simplicity, the real-valued linear system for massive MIMO detection with K transmitting and N receiving antennas is considered as follows

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where the transformation from the complex system model to the real one is straightforward [1]. Here, $\mathbf{H} \in \mathbb{R}^{N\times K}$ is the channel matrix containing independent, identically distributed (i.i.d.) Gaussian fading gains with unit variance and remaining constant over each frame duration, $\mathbf{y} \in \mathbb{R}^N$, $\mathbf{x} \in \mathbb{R}^K$ and $\mathbf{n} \in \mathbb{R}^N$ denote the transmitted signal, the corresponding received signal and the zero-mean additive white Gaussian noise with variance σ_n^2, respectively. Then the problem of massive MIMO detection under maximum a posteriori (MAP) criterion reads

$$\widehat{\mathbf{x}} = \arg\max_{\mathbf{x}\in\mathcal{Q}^K} p(\mathbf{x}|\mathbf{y}), \tag{2}$$

where $\mathcal{Q} = \{\pm 1, \pm 3, ..., \pm\sqrt{M}-1\}$ and M represents the index of the quadrature amplitude modulation (QAM).

In order to estimate $p(\mathbf{x}|\mathbf{y})$, the variational inference algorithm in [2] is introduced to find an approximate distribution $q(\mathbf{x})$ as close to $p(\mathbf{x}|\mathbf{y})$ as possible, which corresponds to

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minimizing the KL-divergence between them

$$\mathrm{KL}(q||p) = \int q(\mathbf{x}) \ln \left\{ \frac{q(\mathbf{x})}{p(\mathbf{x}|\mathbf{y})} \right\} d\mathbf{x}. \tag{3}$$

However, since the posterior $p(\mathbf{x}|\mathbf{y})$ itself is intractable, variational inference instead turns to maximize the ELBO

$$\mathcal{L}(q) = \int q(\mathbf{x}) \ln \left\{ \frac{p(\mathbf{x}, \mathbf{y})}{q(\mathbf{x})} \right\} d\mathbf{x} \tag{4}$$

based on the relationship $\ln p(\mathbf{y}) = \mathcal{L}(q) + \mathrm{KL}(q||p)$, where the tractability of the complete data distribution $p(\mathbf{x}, \mathbf{y}) = p(\mathbf{y}|\mathbf{x})p(\mathbf{x})$ is possible. Intuitively, since $\mathrm{KL}(q||p) \geq 0$, it follows that $\mathcal{L}(q) \leq \ln p(\mathbf{y})$, where the equality holds if and only if the approximated distribution is exactly the same as the true posterior, i.e., $q(\mathbf{x}) = p(\mathbf{x}|\mathbf{y})$. Moreover, with so many choices for $q(\mathbf{x})$, the *mean field theory* framework constrains its distribution by the following partitioned form:

$$q(\mathbf{x}) = \prod_{k=1}^{K} q_k(x_k), \tag{5}$$

so that $q(\mathbf{x})$ becomes tractable. In this way, among all the distributions $q(\mathbf{x})$ that satisfy (5), the one with the maximized $\mathcal{L}(q)$ is desired. Specifically, substituting (5) into (4) gives

$$\begin{aligned} \mathcal{L}(q) &= \int \prod_k q_k \left\{ \ln p(\mathbf{x}, \mathbf{y}) - \sum_k \ln q_k \right\} d\mathbf{x} \\ &= \int q_i \left\{ \int \ln p(\mathbf{x}, \mathbf{y}) \prod_{k \neq i} q_k d\mathbf{x}^{\setminus i} \right\} dx_i - \int q_i \ln q_i dx_i + C \\ &\overset{(a)}{=} \int q_i \ln \widetilde{p}(x_i, \mathbf{y}) \ dx_i - \int q_i \ln q_i \ dx_i + C \\ &= -\mathrm{KL}(q_i(x_i)||\widetilde{p}(x_i, \mathbf{y})) + C, \end{aligned} \tag{6}$$

where the dependence of $q_k(x_k)$ on x_k is omitted for simplicity of notation. Without loss of generality, C denotes an arbitrary constant in the rest of this paper. Here $\mathbf{x}^{\setminus i}$ denotes all the components of $\mathbf{x}$ except x_i, and a new distribution $\widetilde{p}(x_i, \mathbf{y})$ in equality (a) is defined by its logarithm form as

$$\ln \widetilde{p}(x_i, \mathbf{y}) \triangleq \int \ln p(\mathbf{x}, \mathbf{y}) \prod_{k \neq i} q_k d\mathbf{x}^{\setminus i}. \tag{7}$$

From (6), it is clear that maximizing ELBO $\mathcal{L}(q)$ accounts for minimizing $\mathrm{KL}(q_i(x_i)||\widetilde{p}(x_i, \mathbf{y}))$ for $i = 1, ..., K$, which occurs when these two distributions are proportional to each other, i.e.,

$$q_i(x_i)/\widetilde{p}(x_i, \mathbf{y}) = C. \tag{8}$$

Therefore, the i-th optimal posterior solution $q_i^*(x_i)$ can be estimated via $\ln \widetilde{p}(x_i, \mathbf{y})$ as

$$\begin{aligned} \ln q_i^*(x_i) &= \ln \widetilde{p}(x_i, \mathbf{y}) + C = \int \ln p(\mathbf{x}, \mathbf{y}) \prod_{k \neq i} q_k d\mathbf{x}^{\setminus i} + C \\ &= \mathbb{E}_{\setminus i} \big[\ln p(\mathbf{x}, \mathbf{y}) \big] + C, \end{aligned} \tag{9}$$

where a normalization would eliminate the influence of C. This equation (9) formulates the most important update of mean-field VI method, which indicates that $\ln q_i^*(x_i)$ can be attained by taking expectation of $\ln p(\mathbf{x}, \mathbf{y})$ over all factors except q_i. At this point, the i-th detection result can be obtained as follows

$$\widehat{x}_i = \arg \max_{x_i \in \mathcal{Q}} q_i^*(x_i). \tag{10}$$

III. Algorithm Description

For a massive MIMO system with a fixed K, as N goes larger, the channel matrix $\mathbf{H}$ becomes more orthogonal. Theoretically, this is referred to as the *favorable propagation* characteristic [9]. Hence, its corresponding normalized Gram matrix $\mathbf{H}^T\mathbf{H}/N$ would asymptotically approach an identity matrix, i.e.,

$$\lim_{N \to \infty} \frac{\mathbf{H}^T \mathbf{H}}{N} \approx \mathbf{I}_K. \tag{11}$$

Accordingly, we can update the system model in (1) as

$$\mathbf{z} = \mathbf{J}\mathbf{x} + \mathbf{v}, \tag{12}$$

with $\mathbf{z} = \mathbf{H}^T\mathbf{y}/N$, $\mathbf{J} = \mathbf{H}^T\mathbf{H}/N$ and $\mathbf{v} = \mathbf{H}^T\mathbf{n}/N$[1]. Then the complete data distribution $p(\mathbf{x}, \mathbf{y})$ in (9) is converted to $p(\mathbf{x}, \mathbf{z})$. Based on this favorable propagation, we are able to derive the concrete form of equation (9) for detection.

Specifically, let g_i denote the linear combination of $\mathbf{x}^{\setminus i}$ and v_i, $g_i \triangleq \sum_{k \neq i} J_{ik} x_k + v_i$, then z_i is formed in $z_i = J_{ii} x_i + g_i$. Owing to favorable propagation, the value of z_i is dominated by $J_{ii} x_i$, so that z_i can be treated as conditionally independent of $\mathbf{x}^{\setminus i}$ given x_i, denoted as

$$z_i \perp \mathbf{x}^{\setminus i} | x_i, \tag{13}$$

and g_i is therefore treated as the interference term in z_i concerning x_i. Then the likelihood $p(z_i|\mathbf{x})$ can be approximated by $p(z_i|x_i)$, $p(z_i|\mathbf{x}) \approx p(z_i|x_i)$. At the same time, following the mean-field partition in (5), we assume the posterior of $\mathbf{x}$ to be independent Gaussian. Then it is easy to verify that g_i is also Gaussian distributed, i.e., $g_i \sim \mathcal{N}(g_i|\mu_{g_i}, \sigma_{g_i})$. Therefore, as a result of the conditional independence in (13), g_i can be treated as an additive Gaussian noise, leading to the likelihood

$$p(z_i|x_i) \sim \mathcal{N}(z_i|J_{ii} x_i + \mu_{g_i}, \sigma_{g_i}^2). \tag{14}$$

To this end, the log of the complete data distribution in (9) shows the following tractable form:

$$\begin{aligned} \ln p(\mathbf{x}, \mathbf{z}) &= \ln p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = \sum_{k=1}^{K} \ln p(z_k|x_k) + \sum_{k=1}^{K} \ln p(x_k) \\ &= -\sum_{k=1}^{K} \frac{1}{2\sigma_{g_k}^2} (z_k - J_{kk} x_k - \mu_{g_k})^2 + \sum_{k=1}^{K} \ln \left\{ \frac{\mathbb{I}_{x_k \in \mathcal{Q}}}{M^{\frac{1}{2}}} \right\}, \end{aligned} \tag{15}$$

where the prior of $\mathbf{x}$, $p(\mathbf{x})$, also obeys the mean-field constraint but is non-informative:

$$p(\mathbf{x}) = \prod_{k=1}^{K} p(x_k) = \frac{1}{M^{\frac{K}{2}}} \prod_{k=1}^{K} \mathbb{I}_{x_k \in \mathcal{Q}}. \tag{16}$$

Here, $\mathbb{I}_{x_k \in \mathcal{Q}}$ is the indicator function that takes value one if $x_k \in \mathcal{Q}$ and zero otherwise.

Next, according to (9), taking expectations of $\ln p(\mathbf{x}, \mathbf{z})$ over $\prod_{k \neq i}^{K} q_k(x_k)$ entails the i-th optimal posterior $\ln q_i^*(x_i)$. Recall that $\mathbb{E}_{\setminus i}\big[f(x_i)\big] = f(x_i)$, $\mathbb{E}_{\setminus i}\big[f(x_k)\big] = C$ if $k \neq i$, and we have

$$\ln q_i^*(x_i) = \mathbb{E}_{\setminus i}\big[\ln p(\mathbf{x}, \mathbf{z})\big] + C$$

[1] Technically, $\mathbf{v} = \mathbf{H}^T\mathbf{n}/N \sim \mathcal{N}(\mathbf{0}, \frac{\sigma_n^2}{N^2}\mathbf{H}^T\mathbf{H})$. Owing to the property in (11), this can be approximated by $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \frac{\sigma_n^2}{N}\mathbf{I})$.

Algorithm 1 Favorable-Propagation-Exploited Variational Inference Detector (FPE-VI)

Input: $\mathbf{z}, \mathbf{J}, \sigma_v^2 = \sigma_n^2/N, T, \delta$

1: Initialize the posterior as $q_i^{<0>}(s) = \frac{1}{\sqrt{M}}$ for $s \in \mathcal{Q}$.
2: **for** $t = 1$ to T **do**
3: Estimate the mean and variance for x_i by $\mu_i^{<t>} = \sum_{s\in\mathcal{Q}} s \cdot q_i^{<t-1>}(s)$ and $\sigma_i^{2<t>} = \sum_{s\in\mathcal{Q}} s^2 \cdot q_i^{<t-1>}(s) - \mu_i^{<t>2}$.
4: Update the mean and variance for g_i by $\mu_{g_i}^{<t>} = \sum_{k=1,k\neq i}^{K} J_{jk}\mu_k^{<t>}$ and $\sigma_{g_i}^{2<t>} = \sum_{k=1,k\neq i}^{K} J_{jk}^2 \sigma_k^{2<t>} + \sigma_v^2$.
5: Calculate the likelihood $\Lambda_i^{<t>}$ by (19).
6: **if** $t > 1$ **then**
7: $\Lambda_i^{<t>}(s) = \delta\Lambda_i^{<t-1>}(s) + (1-\delta)\Lambda_i^{<t>}(s)$.
8: **end if**
9: Update the posterior $q_i^{<t>}$ by (20).
10: **end for**

Output: $\widehat{x}_i = \arg\max_{s\in\mathcal{Q}} q_i^{<T>}(s)$

$$= \ln \mathbb{I}_{x_i\in\mathcal{Q}} - \frac{1}{2\sigma_{g_i}^2}(z_i - J_{ii}x_i - \mu_{g_i})^2 + C, \quad (17)$$

which implies

$$q_i^*(x_i) \propto \exp\left(\ln \mathbb{I}_{x_i\in\mathcal{Q}} + \Lambda_i(x_i)\right). \quad (18)$$

Here, the log-likelihood of z_i given x_i is defined as $\Lambda_i(x_i)$

$$\Lambda_i(x_i) = -\frac{1}{2\sigma_{g_i}^2}(z_i - J_{ii}x_i - \mu_{g_i})^2. \quad (19)$$

Since $\ln[\mathbb{I}_{x_i\in\mathcal{Q}}(x_i \notin \mathcal{Q})] = \ln 0 = -\infty$, $q_i(x_i \notin \mathcal{Q})$ becomes zero, and the posterior $q_i(x_i)$ naturally has a discrete form. Consequently, letting s denote the symbol belonging to $\mathcal{Q}$, then the normalized posterior is calculated by

$$q_i(s) = \frac{\exp(\Lambda_i(s))}{\sum_{s\in\mathcal{Q}} \exp(\Lambda_i(s))} \quad (20)$$

for all $s \in \mathcal{Q}$. Finally the symbol that gives the largest posterior contributes the result of signal detection as $\widehat{x}_i = \arg\max_{s\in\mathcal{Q}} q_i(s)$.

To summarize, the proposed FPE-VI detection algorithm is outlined in Alg.1, where a superscript $<t>$ is noted for the t-th iteration, T is the total iteration number, and the updating for $i = 1, ..., K$ can be readily implemented in parallel. The updating equations in line 4 derive from the fact that g_i is a linear combination of $\mathbf{x}^{\setminus i}$, and thus $\mu_{g_i} = \sum_{k=1,k\neq i}^{K} J_{jk}\mathbb{E}_{q_k}(x_k)$, $\sigma_{g_i}^2 = \sum_{k=1,k\neq i}^{K} J_{jk}^2 \mathrm{Var}_{q_k}(x_k) + \sigma_v^2$.

At line 6-8, we propose to update the current prior as the posterior from the previous iteration for information supplement since the 2^{nd} iteration. To make it more specific, at $t = 2$, recall by (18) that the previous posterior $q_i^{<1>}(x_i)$ is proportional to $\exp\left(\Lambda_i^{<1>}(x_i)\right)$ when $x_i \in \mathcal{Q}$. Then $\ln q_i^{<1>}(x_i) = \Lambda_i^{<1>}(x_i) + C$. Replace this for the prior $p(\mathbf{x})$ in (15) and we get $q_i^{<2>}(x_i) \propto \exp\left(\ln q_i^{<1>}(x_i) + \Lambda_i^{<2>}(x_i)\right) \propto \exp\left(\Lambda_i^{<1>}(x_i) + \Lambda_i^{<2>}(x_i)\right)$. Carrying out this recursively leads to $q_i^{<t>}(x_i) \propto \exp\left(\delta^{t-1}\Lambda_i^{<1>}(x_i) + \delta^{t-2}\Lambda_i^{<2>}(x_i) + ... + \delta\Lambda_i^{<t-1>}(x_i) + (1-\delta)\Lambda_i^{<t>}(x_i)\right)$, where a weighted form with the parameter $0 < \delta < 1$ is introduced to control how much the updated prior is considered at each iteration. This can be interpreted as the damping technique [10], which is often adopted for convergence consideration. The parameter δ is the corresponding damping factor. Moreover, we remark that the practical implementation of the proposed FPE-VI is similar to that of the channel hardening exploited message passing (CHEMP) algorithm in [5]. However, these two detection schemes are derived from totally different perspectives. The proposed FPE-VI detector is the approximate inference method with specific $\mathcal{L}(q)$ to optimize. From this point of view, CHEMP happens to be the special case of variational inference detection where $\mathbf{x}$ is assumed to be independent Gaussian.

IV. Convergence and Complexity Analysis

A. Convergence Analysis

We now examine the convergence of FPE-VI based on its linear counterpart, where the nonlinear estimations in relation to (20) can be eliminated. Note that the Gaussian random variable x_i is a linear transformation of g_i, i.e., $x_i = (z_i - g_i)/J_{ii}$. Then the mean and variance for x_i can also be estimated by $\mu_i = (z_i - \mu_{g_i})/J_{ii}$ and $\sigma_i^2 = \sigma_{g_i}^2/J_{ii}^2$, respectively. Rewriting them in matrix forms gives

$$\boldsymbol{\mu}^{<t>} = \mathbf{D}^{-1}(\mathbf{z} - \boldsymbol{\mu}_g^{<t-1>}); \boldsymbol{\sigma}^{2<t>} = [\mathbf{D}^{(2)}]^{-1}\boldsymbol{\sigma}_g^{2<t-1>}. \quad (21)$$

Here $\mathbf{D}$ is a diagonal matrix with the diagonal elements of $\mathbf{J}$, and $\mathbf{D}^{(2)}$ denotes the element-wise square of $\mathbf{D}$. Clearly, by iterations, the updates at line 4 of Alg.1 can be expressed as

$$\begin{aligned} \boldsymbol{\mu}_g^{<t>} &= \mathbf{J}\boldsymbol{\mu}^{<t-1>} - \mathbf{D}\boldsymbol{\mu}^{<t-1>} = \mathbf{E}\boldsymbol{\mu}^{<t-1>}, \\ \boldsymbol{\sigma}_g^{2<t>} &= \mathbf{J}^{(2)}\boldsymbol{\sigma}^{2<t-1>} - \mathbf{D}^{(2)}\boldsymbol{\sigma}^{2<t-1>} + \boldsymbol{\sigma}_v^2 \\ &= \mathbf{E}^{(2)}\boldsymbol{\sigma}^{2<t-1>} + \boldsymbol{\sigma}_v^2 \end{aligned} \quad (22)$$

with $\boldsymbol{\sigma}_v^2 = \sigma_n^2 \mathbf{1}_{K\times 1}/N$ and $\mathbf{E} = \mathbf{J} - \mathbf{D}$. The overall linear FPE-VI therefore implements (22) and (21) alternatively.

Theorem 1. *For massive MIMO systems with fixed antenna ratio $\alpha = N/K$, the mean of the linear FPE-VI detection converges to the solution of ZF detection if*

$$\alpha > \frac{1}{(\sqrt{2}-1)^2} \approx 5.83. \quad (23)$$

Proof: By combing (21) and (22), the iteration of the mean vector for $\mathbf{x}$ can be derived by

$$\boldsymbol{\mu}^{<t>} = \mathbf{D}^{-1}(\mathbf{z} - \mathbf{E}\boldsymbol{\mu}^{<t-1>}) = -\mathbf{D}^{-1}\mathbf{E}\boldsymbol{\mu}^{<t-1>} + \mathbf{D}^{-1}\mathbf{z}. \quad (24)$$

Then, according to convergence theory [11], as long as the spectral radius of $-\mathbf{D}^{-1}\mathbf{E}$ is less than 1, i.e.,

$$\rho(-\mathbf{D}^{-1}\mathbf{E}) < 1, \quad (25)$$

$\boldsymbol{\mu}$ will be convergent to the following solution

$$\boldsymbol{\mu}^* = (\mathbf{I} + \mathbf{D}^{-1}\mathbf{E})^{-1}\mathbf{D}^{-1}\mathbf{z} = (\mathbf{D} + \mathbf{E})^{-1}\mathbf{z} = \mathbf{J}^{-1}\mathbf{z}, \quad (26)$$

which is just the result of ZF detection.

Next, with respect to the spectrum radius condition in (25), it has the form

$$\left|1 - \frac{1}{N}\lambda(\mathbf{H}^T\mathbf{H})\right| < 1, \quad (27)$$

and can be further expressed by

$$\lambda(\mathbf{H}^T\mathbf{H}) < 2N, \quad (28)$$

which imposes the requirement that the largest eigenvalue of the matrix $\mathbf{H}^T\mathbf{H}$ should be less than $2N$. According to *random matrix theory* [12], the maximum eigenvalue of $\mathbf{H}^T\mathbf{H}$ converges to $\lambda_{max} = N(1+\sqrt{K/N})^2$, when the number of antennas K and N approach infinity. Hence, it is easy to arrive at (23). ■

Theorem 2. *For massive MIMO systems, the variance of the linear FPE-VI detection converges to*

$$\boldsymbol{\sigma}^{2*} = \widetilde{\mathbf{J}}^{-1}\boldsymbol{\sigma}_v^2 \tag{29}$$

with $\widetilde{\mathbf{J}} \triangleq \mathbf{D}^{(2)} - \mathbf{E}^{(2)}$, *if the matrix* $\mathbf{H}^T\mathbf{H}$ *is diagonally dominant.*

Proof: Similar to the proof for Theorem 1, the iteration of the variance vector $\boldsymbol{\sigma}$ can be formulated as

$$\boldsymbol{\sigma}^{2<t>} = [\mathbf{D}^{(2)}]^{-1}\mathbf{E}^{(2)}\boldsymbol{\sigma}^{2<t-1>} + [\mathbf{D}^{(2)}]^{-1}\boldsymbol{\sigma}_v^2. \tag{30}$$

As a result, $\boldsymbol{\sigma}$ would converge to

$$\boldsymbol{\sigma}^{2*} = \left\{\mathbf{I} - [\mathbf{D}^{(2)}]^{-1}\mathbf{E}^{(2)}\right\}^{-1}[\mathbf{D}^{(2)}]^{-1}\boldsymbol{\sigma}_v^2 = \left\{\mathbf{D}^{(2)} - \mathbf{E}^{(2)}\right\}^{-1}\boldsymbol{\sigma}_v^2 \triangleq \widetilde{\mathbf{J}}^{-1}\boldsymbol{\sigma}_v^2 \tag{31}$$

if the convergence matrix $\mathbf{I} - [\mathbf{D}^{(2)}]^{-1}\mathbf{E}^{(2)} \triangleq \mathbf{I} - \widetilde{\mathbf{E}}^{(2)}$ is diagonally dominant [11]. In (31) we define $\widetilde{\mathbf{J}} \triangleq \mathbf{D}^{(2)} - \mathbf{E}^{(2)}$, which is made up of the components of $\mathbf{J}^{(2)}$ except that those non-diagonal ones become their corresponding opposite numbers, i.e., $\widetilde{J}_{ij} = J_{ij}^2$, if $i = j$; else $\widetilde{J}_{ij} = -J_{ij}^2$. Since $[\mathbf{D}^{(2)}]^{-1}\mathbf{E}^{(2)}$ divides each row of $\mathbf{E}^{(2)}$ by J_{jj}^2, the convergence matrix has the form

$$[\mathbf{I} - \widetilde{\mathbf{E}}^{(2)}]_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -\dfrac{J_{ij}^2}{J_{jj}^2}, & \text{if } i \neq j. \end{cases}$$

Then, the condition for it to be a diagonally dominant matrix is

$$\sum_{j, j\neq i}\left|\frac{J_{ij}}{J_{jj}}\right|^2 < 1, \ \forall i, j = 1, ..., K. \tag{32}$$

Considering that the diagonal elements J_{jj} approach to 1 when the number of N is large, (32) can be satisfied if

$$\sum_{j, j\neq i}|J_{ij}| = \sum_{j, j\neq i}\left|\frac{\mathbf{h}_i^T\mathbf{h}_j}{N}\right| < 1, \ \forall i, j = 1, ..., K. \tag{33}$$

This is in fact the condition for the matrix $\mathbf{H}^T\mathbf{H}$ to be diagonally dominant, which has been analyzed in [13]. ■

To describe the extent of how diagonally dominant the matrix $\mathbf{H}^T\mathbf{H}$ is, we quantify this in terms of the antenna ratio α as [13] does. Before that, with respect to the condition in (33), we make it more flexible by relaxing the inequality to a threshold β no smaller than 1,

$$\sum_{j, j\neq i}|J_{ij}| < \beta, \ \forall i, j = 1, ..., K; \beta \geq 1. \tag{34}$$

The required minimum α for $\mathbf{H}^T\mathbf{H}$ to be diagonally dominant given β and N is presented in Table I. The corresponding β for $\alpha > 1/(\sqrt{2}-1)^2$ under systems with $N = 32$, $N = 48$, and $N = 64$ are $\beta < 1.68$, 2.13, and 2.50, respectively, where the matrix $\mathbf{H}^T\mathbf{H}$ can still be treated as diagonally dominant. What is more, note that when the condition is fulfilled, the matrix $\widetilde{\mathbf{J}}$ is also diagonally dominant and can reduce to $\mathbf{D}^{(2)}$. Therefore the variance vector for the interference term $\boldsymbol{\sigma}_g^2$ would converge to $\mathbf{D}^{(2)}[\mathbf{D}^{(2)}]^{-1}\boldsymbol{\sigma}_v^2 = \boldsymbol{\sigma}_v^2$. This inspires a simplification on the computation of $\boldsymbol{\sigma}_g^2$: directly set $\boldsymbol{\sigma}_g^2 = \boldsymbol{\sigma}_v^2$, leading to further complexity reduction on FPE-VI algorithm.

TABLE I
REQUIRED MINIMUM ANTENNA RATIO α

	$\beta=1$	$\beta=3$	$\beta=5$	$\beta=7$	$\beta=9$
$N=32$	9.2282	3.4032	2.0836	1.5042	1.1761
$N=48$	11.6135	4.2107	2.5716	1.8510	1.4459
$N=64$	13.6334	4.8918	2.9807	2.1433	1.6732

TABLE II
THE NUMBER OF MULTIPLICATIONS IN EACH ITERATION FOR $i = 1, ..., K$

Estimate $\mu_i^{<t>}$	$K\sqrt{M}$	$\sigma_i^{2<t>}$	$K\sqrt{M}+K+\sqrt{M}$
Update $\mu_{g_i}^{<t>}$	K^2	$\sigma_{g_i}^{2<t>}$	$2K^2$
Calculate $\Lambda_i^{<t>}$	$3K\sqrt{M}$	Damping	$2K\sqrt{M}$
Update $q_i^{<t>}$	$2K\sqrt{M}$	**Sum**	$3K^2+9K\sqrt{M}+K+\sqrt{M}$
Order		$O(K^2)$	

Remark 1. $\alpha > 1/(\sqrt{2}-1)^2$ *is generally sufficient for both the mean and variance of linear FPE-VI to be convergent.*

B. Complexity Analysis

In this section, the computational complexity of the proposed FPE-VI detection is discussed. For those procedures implemented in each iteration, Table IV-B lists all the related multiplication times. Here are some remarks about this table: Firstly, for some results that can be stored to reuse, the computations on them are counted only once. A numerical division operation is assumed to have the same complexity as a multiplication. Moreover, the required multiplication for a diagonal matrix product is reduced owing to the diagonal structure. For example, $\mathbf{H}_{N\times K}\mathbf{D}_{K\times K}$ needs only $N \times K$ times multiplications. As for the preprocessing to obtain the system in (12), the complexity is of order $O(NK^2)$ due to the calculation of $\mathbf{H}^T\mathbf{H}$. Therefore, the overall complexity of the FPE-VI detection is dominated by the preprocessing step and can be written as of order $O(NK^2 + K^2T)$. Once the transfer to the system in (12) is done, the complexity of FPE-VI is independent of the number of receiving antennas N and thus the antenna ratio α, since it operates on the converted Gram matrix $(\mathbf{H}^T\mathbf{H})_{K\times K}$. On the whole, for a fixed K, the larger the antenna ratio α is, the more noticeable the favorable propagation phenomenon is, and consequently the more advantages the FPE-VI detection can take from both the complexity and performance.

V. FURTHER ENHANCEMENT

In the sequel, we go through the ELBO of FPE-VI for further improvement. The ELBO $\mathcal{L}(q)$ in (4) has the following form:

$$\mathcal{L}(q) = \underbrace{\int q(\mathbf{x})\ln p(\mathbf{x})d\mathbf{x}}_{\mathcal{L}_a} + \underbrace{\int q(\mathbf{x})\ln p(\mathbf{z}|\mathbf{x})d\mathbf{x}}_{\mathcal{L}_b} + \mathbb{H}(q), \tag{35}$$

where $\mathbb{H}(q) = -\int q(\mathbf{x})\ln q(\mathbf{x})d\mathbf{x}$ is the entropy of $q(\mathbf{x})$. Specifically, the first term $\mathcal{L}_a$ is a constant

$$\mathcal{L}_a = \int q(\mathbf{x}) \ln \prod_{k=1}^{K} \frac{\mathbb{I}_{x_k \in \mathcal{Q}}}{M^{\frac{1}{2}}} d\mathbf{x} = -\frac{K}{2} \ln M, \tag{36}$$

with the second term

$$\begin{aligned}\mathcal{L}_b &= \int \prod_k q_k \sum_k \ln p(z_k|x_k) d\mathbf{x} = \sum_k \int q_k \ln p(z_k|x_k) dx_k \\ &= \sum_k \int q_k \left(\ln \frac{1}{\sqrt{2\pi}\sigma_{g_k}} - \frac{(z_k - J_{kk}x_k - \mu_{g_k})^2}{2\sigma_{g_k}^2} \right) dx_k \\ &= -\frac{1}{2} \ln(2\pi)^K |\mathbf{\Sigma}_g| + \sum_k \mathbb{E}_{q_k}[\Lambda_k]. \end{aligned} \tag{37}$$

Here $\mathbf{\Sigma}_g$ is a diagonal matrix whose diagonal is $[\sigma_{g_1}^2, ..., \sigma_{g_K}^2]^T$. Adding up these terms leads to

$$\mathcal{L}(q) = -\frac{1}{2} \ln |\mathbf{\Sigma}_g| + \sum_k \mathbb{E}_{q_k}[\Lambda_k] + \mathbb{H}(q) + C, \tag{38}$$

which monotonically increases by iterations.

So far for FPE-VI, the initial probability of $\mathbf{x}$ is set as a uniform distribution

$$p(x_k) = \frac{1}{\sqrt{M}}, \ k = 1, ...K, \tag{39}$$

corresponding to the value of $\mathcal{L}_a = -(K \ln M)/2$. Clearly, a more reasonable initial prior would contribute to a larger $\mathcal{L}_a$, thus improving the convergence at the beginning. Since the posterior is discrete-Gaussian-distributed, we might as well choose the prior of $\mathbf{x}$ as a discrete Gaussian distribution either. To this end, a remained question is how to determine the mean $\bar{\boldsymbol{\mu}}$ and variance $\bar{\sigma}^2$ of this initial distribution $\widehat{p}(\mathbf{x})$:

$$\widehat{p}_k(s) = \frac{\exp\left(-\frac{1}{2\bar{\sigma}^2}(s - \bar{\mu}_k)^2\right)}{\rho_{\mathcal{Q}}(\bar{\mu}_k, \bar{\sigma}^2)}, \tag{40}$$

where $\rho_{\mathcal{Q}}(\bar{\mu}_k, \bar{\sigma}^2) \triangleq \sum_{s \in \mathcal{Q}} \exp\left(-\frac{1}{2\bar{\sigma}^2}(s - \bar{\mu}_k)^2\right)$ is a positive scalar to ensure a probability distribution. Note that z_i is treated as the combination of two terms: $z_i = J_{ii}x_i + g_i$. Therefore, at the initialization stage, in order to give a rough initial value about the mean $\bar{\boldsymbol{\mu}}$, g_i can be ignored, leading to

$$\bar{\boldsymbol{\mu}} = \mathbf{D}^{-1}\mathbf{z}. \tag{41}$$

This inversion needs only K times divisions thanks to the diagonal structure of $\mathbf{D}$. As for the choice of $\bar{\sigma}^2$, notice that $\mathcal{L}_a$ becomes

$$\begin{aligned}\widehat{\mathcal{L}}_a(q) &= \sum_{\mathcal{Q}^K} \prod_k q_k \ln \prod_k \widehat{p}_k = \sum_k \sum_{s \in \mathcal{Q}} q_k(s) \ln \widehat{p}_k(s) \\ &= \sum_k \sum_{s \in \mathcal{Q}} q_k(s) \left[-\ln \rho_{\mathcal{Q}}(\bar{\mu}_k, \bar{\sigma}^2) - \frac{1}{2\bar{\sigma}^2}(s - \bar{\mu}_k)^2 \right] \\ &\triangleq -\sum_k \left[\ln \rho_{\mathcal{Q}}(\bar{\mu}_k, \bar{\sigma}^2) + \frac{1}{2} \cdot \frac{\gamma_k(q_k)}{\bar{\sigma}^2} \right], \end{aligned} \tag{42}$$

where we have defined $\gamma_k(q_k) \triangleq \sum_{s \in \mathcal{Q}} (s - \bar{\mu}_k)^2 q_k(s)$. Intuitively, as $\bar{\sigma}^2 \to 0$, it follows that $\widehat{p}_k(s \neq s_{\bar{\mu}_k}) \to 0$, where $s_{\bar{\mu}_k}$ is the symbol closest to $\bar{\mu}_k$, resulting in a negative infinite value of $\widehat{\mathcal{L}}_a(q)$. When $\bar{\sigma}^2 \to +\infty$, this comes to the uniform case and thus $\widehat{\mathcal{L}}_a(q)$ approaches the $-(K \ln M)/2$ in (36). Besides, $\ln \rho_{\mathcal{Q}}(\bar{\mu}_k, \bar{\sigma}^2)$ is monotonically increasing with respect to $\bar{\sigma}^2 > 0$ due to its positive partial derivative $\partial \ln \rho_{\mathcal{Q}} / \partial \bar{\sigma}^2 = (1/\rho_{\mathcal{Q}}) \sum_{s \in \mathcal{Q}} \frac{(s - \bar{\mu}_k)^2}{2\bar{\sigma}^4} e^{-\frac{(s - \bar{\mu}_k)^2}{2\bar{\sigma}^2}} > 0$, and the decreasing monotonicity of $\gamma_k(q_k)/2\bar{\sigma}^2$ is obvious. Hence, when $e^{\frac{-(s - \bar{\mu}_k)^2}{2\bar{\sigma}^2}} / \rho_{\mathcal{Q}} = q_k(s)$ for $\forall s \in \mathcal{Q}$, $\widehat{\mathcal{L}}_a(q)$ reaches its maximum. This solution is intractable, but using numerical method to find $\bar{\sigma}^2$ that gives nearly satisfactory performance is possible and clearly a more reasonable way.

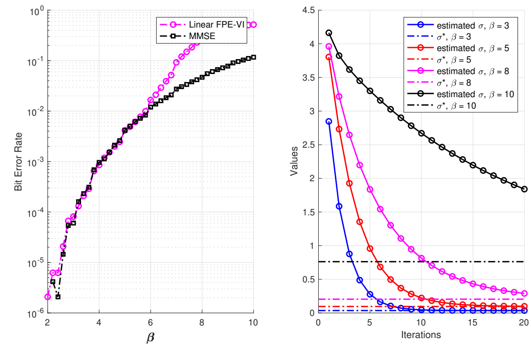

Fig. 1. Investigations on the linear version of the proposed FPE-VI, where 16-QAM modulation is used with $N = 64$, Eb/N0 =10 dB. The left picture illustrates the performance comparison between the linear FPE-VI and MMSE versus β, and the right shows the difference between the estimated variance by linear FPE-VI and the variance σ^* in (29) versus iterations.

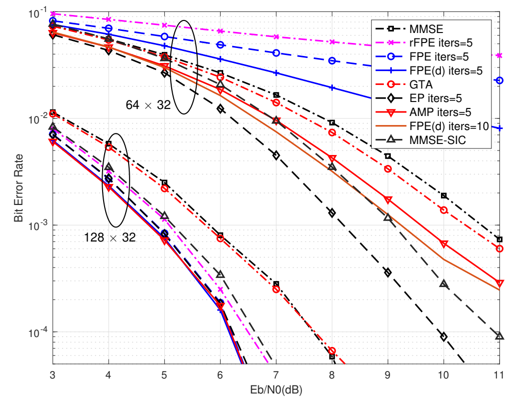

Fig. 2. Performance comparison between the proposed VI detectors, MMSE and some other inference methods for 64×32 and 128×32 massive MIMO systems using 16-QAM.

VI. SIMULATION RESULTS

This section presents some numerical results for the proposed VI detectors, compared to the traditional MMSE detector and some inference algorithms. Explorations on the linear FPE-VI according to aforementioned analysis is given in Fig.1. In the left picture, we compare the linear FPE-VI and MMSE detector versus the parameter β in the relaxed condition (34). Before the β reaches 6, it is still possible for the linear FPE-VI to converge to the MMSE detection. Note that the larger the β is, the less likely $\mathbf{J}$ is going to be diagonally dominant. Their corresponding antenna ratios computed by the method in [13] can be found by looking up the Table I. The right picture presents the difference between the estimated variance by linear FPE-VI and the variance σ^* in (29) at some specific β. When β increases to 8, this gap rises and grows considerably large when $\beta = 10$. On the whole, the condition on β with respect to the mean is more strict than that for the variance.

In Fig.2, the proposed FPE-VI detectors are compared with some other inference methods, including the GTA detection in [3], the Gaussian AMP algorithm in [4], and the EP in [7]. The damping factors are set as $\delta = 0.4$ for 64×32 system

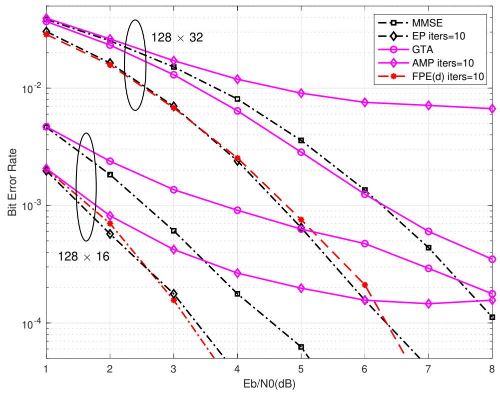

Fig. 3. Performance comparison between the proposed FPE-VI detector, MMSE and some other inference methods for 128×32 and 128×16 massive correlated MIMO systems using 16-QAM.

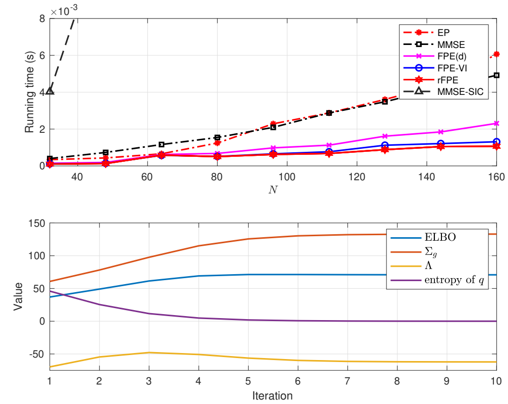

Fig. 4. Running time of the proposed FPE-VI detectors, MMSE, and EP for 16QAM massive MIMO system, where the antenna ratio $\alpha = 2$ and all the iterative methods are implemented by 5 iterations (upper). The evolution of the ELBO value in FPE-VI over iterations, conducted under the 64×32 massive MIMO system using 16QAM, with fixed Eb/N0=8dB (lower).

and $\delta = 0.3$ for the 128×32 one. Under 64×32 scheme, all the FPE-VI detectors suffer from disadvantages in the antenna ratio. Among them, the FPE-VI initialized with the discrete Gaussian distribution, FPE(d), significantly outperforms the pure FPE-VI with the same number of iterations. Meanwhile, the reduced FPE-VI with $\boldsymbol{\sigma}_g^2 = \boldsymbol{\sigma}_v^2$, denoted by rFPE, shows some minor performance loss. Still, the FPE-VI(d) struggles to outperform the linear MMSE by extending the total iteration to 10. As for the 128×32 scheme, a faster convergence of FPE-VI can be seen by this increased antenna ratio, in addition to its comparable performance against the EP detector, while it seems difficult for the GTA to overcome the performance bottleneck of the MMSE detection. Moreover, under this circumstance, it is possible for the proposed FPE-VI detectors to outperform the nonlinear MMSE-SIC detection.

Fig. 3 further shows the BER of the proposed FPE algorithm under correlated channels in 128×32 and 128×16 MIMO system with 16-QAM. Without loss of generality, we follow the model in [14] with a normalized correlation coefficient ρ to adjust the degree of correlation. Note that a totally uncorrelated scenario corresponds to $\rho = 0$ while a fully correlated scenario implies $\rho = 1$. Here, we set $\rho = 0.02$, which results in weak correlated channels. As can be seen, the proposed FPE-VI still shows a comparable performance as the EP detection while the other two methods already begin to deteriorate under this weak correlated case. The upper of Fig.4 plots the running time of the related detectors with antenna ratio $\alpha = 2$. The nonlinear MMSE-SIC detection requires the most running time due to its obligation in successive implementation. The reduced FPE-VI shortens the running time while even the FPE-VI(d) still implements faster than the EP and MMSE. Therefore, the advantage of FPE-VI detector in terms of time complexity is significant. The ELBO value of FPE-VI is also presented in the lower of Fig.4, where each individual term in (35) is shown as well. Clearly, the ELBO is monotonically increasing over the iterations, which verifies the convergence of FPE-VI.

VII. Conclusion

In this paper, the favorable-propagation-exploited variational inference (FPE-VI) algorithm is proposed for low-complexity massive MIMO detection. The convergence of its linear counterpart is demonstrated with an antenna ratio α greater than $1/(\sqrt{2}-1)^2$. Moreover, according to the related analysis on evidence lower bound, its initial distribution is also optimized by adopting discrete Gaussian distribution, therefore achieving a better performance-complexity trade-off.

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