A New Method of Integer Parameter Estimation in Linear Models With Applications to GNSS High Precision Positioning

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***Abstract*—The high precision positioning of global navigation satellite systems (GNSS), which essentially corresponds to solving the integer least-squares (ILS) problem in the integer phase ambiguity estimation, has emerged as a key problem for the development of industrial internet-of-things (IIoT). In this paper, a novel paradigm for solving the ILS problem is introduced to the integer phase ambiguity estimation. Different from the traditional paradigm of ILS which only involves one parameter to characterize the trade-off between performance and complexity, the proposed ILS paradigm entails two parameters named as the initial searching size $K \geq 1$ and the standard deviation $\sigma > 0$, thus introducing extra degrees of freedom to facilitate the system trade-off. Based on it, explicit analysis can be carried out for a mathematically tractable trade-off, where great potentials could be further exploited for the high precision positioning of GNSS. The equivalent searching algorithm (ESA) is proposed, which achieves the same performance as the classic Fincke-Pohst strategy in sphere decoding (SD) but with tractable complexity measured by the number of visited nodes in the searching stage. Moreover, the candidate protection mechanism is given to further upgrade the equivalent searching algorithm, which makes it not only an optimal but also a sub-optimal ILS estimator given the flexible setup of K.**

***Index Terms*—Integer parameter estimation, integer phase ambiguity estimation, high precision positioning of GNSS, integer least-squares (ILS) problem.**

I. Introduction

NOWADAYS, the development of global navigation satellite systems (GNSS) has become indispensable for various applications in the industrial internet-of-things (IIoT), which enables efficient and reliable positioning anytime anywhere [1]–[4]. Generally, the equipment calculates its position according to received signals from satellites orbiting the earth, and the accuracy of positioning in IIoT varies flexibly from meters to millimeters [5], [6]. In principle, since the noise level (i.e., the standard deviation of error) of the RF carrier phase measurement is significantly less than those of other measurements, the high precision positioning of GNSS for IIoT chiefly relies on the accurate tracking of the carrier phase of modulated signals [7], [8]. Typically, the measurement of carrier phase consists of two parts — a fraction part and an integer part, where the former could be easily captured by the usage of phase locked loops (PLLs) [9]. In this condition, how to determine the integer part of the carrier phase (usually known as integer phase ambiguity) within a short observation time span has emerged as the key for the millimeter level accuracy [10]. Meanwhile, in order to further eliminate the satellite and receiver clock offsets, the double-difference carrier phase measurement has been well accepted, which restricts the problem of the phase resolution into exploiting the integer nature of the double-difference phase ambiguities [11], [12]. Essentially, this corresponds to solving the integer least-squares (ILS) problem, which has wide applications in various applications like magnetic resonance imaging (MRI), interferometric synthetic aperture radar (InSAR) and cryptography.

With respect to the double-difference integer phase ambiguity estimation, the most success approach so far is the least-squares ambiguity decorrelation adjustment (LAMBDA) proposed by Teunissen [13], [14]. By providing the unbiased estimates of integer phase ambiguities, LAMBDA is not only widely applied in instantaneous ambiguity resolution for GNSS [15], but also is used for the real-time kinematic (RTK) positioning [16]. Essentially, LAMBDA accounts for solving an ILS problem, which can be interpreted by two operation stages — "decorrelation" and "search". By decorrelating the integer ambiguities based on the related variance-covariance matrix, LAMBDA effectively reduces the integer searching space, which results in an efficient discrete searching process thereafter [17]. Based on it, the lattice reduction technique like Lenstra-Lenstra-Lovász (LLL) reduction from the field of lattice theory was introduced to generate an equivalent set but with less correlated integer ambiguities [18]–[20], thus leading to a more efficient ILS

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estimator [21]. Other modified decorrelation works to facilitate the following searching process can be found in [22]–[26].

On the other hand, a lot of works have also been made to speed up the searching process at the second state of LAMBDA. Specifically, the searching process of LAMBDA initially follows the Fincke-Pohst (FP) strategy in lattice decoding [27], which enumerates all the candidate integer vectors within a sphere radius in the space. In [28], a shrinking strategy was proposed, where the searching radius is dynamically updated as soon as a valid candidate integer vector is found. This is actually accordance with the Schnorr-Euchner (SE) strategy in lattice theory [29] while other modifications about the searching stage can be found in [30]–[34]. Meanwhile, with the development of IIoT, the system dimension goes up as the increase in satellite availability and transmitted signals improves the number of ambiguity parameters. For this reason, a number of modification strategies have been given to lower the complexity cost of GNSS positioning [35], [36].

In this condition, reasonable suboptimal solutions based on the searching process become alternatives but it is not clear how the complexity reduction of the searching stage characterizes the performance loss correspondingly. In other words, the trade-off between the searching performance and complexity in addressing the ILS problem is still an open issue. This is a fundamental question in the field of ILS but has not been well explained for a long time. For example, once the searching radius in the searching process is set to guarantee the optimal output solution for the ILS estimator, it is normally hard to track the number of visiting nodes along the tree-like traversal, rendering an untractable complexity cost. Consequently, the unknown trade-off between the performance and complexity severely limits the potential to be further exploited in this area even though a few works have been given to yield the suboptimal solutions in practice.

In this paper, to achieve the high precision positioning of GNSS, a new paradigm of solving the ILS problem is introduced to the integer phase ambiguity estimation in high-dimensional systems. Specifically, the classic ILS problem is reformulated into a new formation but with two controllable parameters: the initial searching size $K \geq 1$ and the standard deviation $\sigma > 0$. Different from the conventional paradigm of the ILS problem with only one tractable parameter (i.e., searching radius $\chi > 0$) to regulate, extra degrees of freedom in characterizing the trade-off between performance and complexity can be obtained. Based on it, the equivalent searching algorithm (ESA) is proposed to achieve the tractable trade-off between performance and complexity, where the complexity by means of the number of visited nodes along the searching process for a given specific sphere radius can be clearly upper bounded. Moreover, with the sphere radius being characterized by K and σ as $\chi = \sigma\sqrt{2\ln K}$, we further update the proposed algorithm to a suboptimal ILS estimator by designing a searching mechanism named as *candidate guard* so that any attempt of complexity reduction in the accessible expense of performance loss can be carried out freely by simply tuning K with a fixed σ. By doing this, the proposed ESA turns out be an optimal scheme when K is large enough and becomes suboptimal when K is limited, which fully takes advantages of the complexity cost to efficiently yield solutions rather than restart the searching again and again with a larger size of sphere radius. In this way, a more flexible positioning for GNSS according to the requirements of time consuming and positioning accuracy is achieved.

To summarize, we advance the state of the art of solving the double-difference integer phase ambiguity estimation in GNSS in the following several fronts.

- A new paradigm of solving the ILS problem is introduced to integer phase ambiguity estimation in GNSS, where extra freedom is introduced in characterizing the trade-off between performance and complexity.
- Under the new paradigm, the equivalent searching algorithm is proposed to realize the tractable trade-off, where the sphere radius $\chi = \sigma\sqrt{2\ln K}$ corresponds to the number of visited nodes upper bounded by $|S| \leq nK$, where n is the system dimension.
- The mechanism *candidate protection* is proposed to make the equivalent searching algorithm more flexible, where suboptimal and optimal solutions will be outputted respectively according to different K with a fixed σ.

The rest of this paper is organized as follows. Section II briefly reviews the basics of carrier-phase measurements in GNSS, and introduces the related ILS system model based on the integer phase ambiguity in the double-difference carrier-phase measurement. In Section III, we introduce the new paradigm to reformulate the ILS problem in a new formation, and we show that such a new paradigm is able to interpret the ILS estimator more tractably in both the perspectives of performance and complexity. Then, based on the paradigm of ILS, the equivalent searching algorithm is proposed in Section IV for a tractable system trade-off. After that, the mechanism of candidate protection is proposed for the equivalent searching algorithm to output suboptimal solutions for the consideration of time consuming. Finally, simulation results are given in Section VI and the paper is concluded in Section VII.

Notation: Matrices and column vectors are denoted by upper and lowercase boldface letters, and the transpose, inverse, pseudoinverse of a matrix $\mathbf{B}$ by $\mathbf{B}^T$, $\mathbf{B}^{-1}$, and $\mathbf{B}^\dagger$, respectively. We use $\mathbf{b}_i$ for the ith column of the matrix $\mathbf{B}$, $b_{i,j}$ for the entry in the ith row and jth column of the matrix $\mathbf{B}$. The operator $\lceil x \rfloor$ denotes rounding to the integer closest to x. If x is a complex number, $\lceil x \rfloor$ rounds the real and imaginary parts separately. Finally, in this paper, the complexity of searching algorithms is evaluated by the number of visited nodes (i.e., $|S|$) during the searching along the tree traversal. Meanwhile, the computational complexity is measured by the number of arithmetic operations (additions, multiplications, comparisons, etc.).

II. Preliminaries

In this section, we introduce the background and mathematical tools, which are necessary to describe and analyze the proposed ILS estimator under the new paradigm for the high precision positioning in GNSS.

A. The Carrier-Phase Measurement in GNSS

Basically, the carrier phase measurement Φ can be represented as

$$\Phi = \|\mathbf{r} - \mathbf{R}\| + c(dt - dT) + \lambda\mathbf{N} + \boldsymbol{\epsilon}, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean distance and

- $\mathbf{r}$ is the unknown receiver antenna position vector at signal reception time,
- $\mathbf{R}$ is the given satellite antenna position vector at signal transmission time,
- c is the speed of light in vacuum,
- cdt is the receiver clock range offset,
- cdT is the satellite clock range offset,
- λ is the carrier wavelength of the signal,
- $\mathbf{N}$ is the carrier phase ambiguity.

Here the term $\boldsymbol{\epsilon}$ accounts for the carrier phase measurement noise and biases such as satellite ephemeris errors, tropospheric and ionospheric delays, as well as ranging errors caused by multipath.

Generally, it is very often to differentiate the carrier phase measurements between satellites and between receivers to remove the satellite and receiver clock offsets. This leads to the double-difference carrier phase observation model given by

$$DD\Phi = DD\|\mathbf{r} - \mathbf{R}\| + \lambda DD\mathbf{N} + DD\boldsymbol{\epsilon}, \tag{2}$$

where DD represents the double-difference operator and the ambiguity-term $DD\mathbf{N}$ is known to be integer-valued [31]. Following most of related works, due to the consideration of simplicity we restrict the setup in the following to the two-receiver situation.[1] This actually has no impact upon the general applicability of the proposed method since the method is independent of the number of receivers [37]. From (2), the linearized double-difference observation equations are collected in the following linear system of equations [13]

$$\mathbf{f} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{b} + \mathbf{e} \tag{3}$$

where

- $\mathbf{f} \in \mathbb{R}^m$ is the vector of observed minus computed double-difference carrier phase measurements,
- $\mathbf{x} \in \mathbb{Z}^n$ is the vector of unknown integer double-difference ambiguities,
- $\mathbf{b} \in \mathbb{R}^p$ is the vector that contains the increments of the unknown baseline coordinates,
- $\mathbf{e} \in \mathbb{R}^n$ is the vector of unmodelled errors,
- $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the design matrix for the ambiguity terms,
- $\mathbf{B} \in \mathbb{R}^{m\times p}$ is the design matrix for baseline coordinates.

Here, m denotes the number of dimensions of the observations by double-difference carrier phase measurements, n is the number of ambiguities. The number of baseline coordinates p is minimally 3, which is the case when both the receivers are stationary, but will be a multiple of 3 when the second receiver is moving. Generally, it is assumed $m \geq n + p$, and the $m \times (n + p)$ design matrix $[\mathbf{A}\ \mathbf{B}]$ is assumed to have full rank equaling $(n + p)$, which means enough observations during multiple epochs have been made to determine $\mathbf{x}$ and $\mathbf{b}$. This ensures the uniqueness of the solution for the mixed least-squares problem in the following

$$(\mathbf{x}_{\text{optimal}}, \mathbf{b}_{\text{optimal}}) = \underset{\mathbf{x}\in\mathbb{Z}^n, \mathbf{b}\in\mathbb{R}^p}{\operatorname{argmin}} \|\mathbf{f} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{b}\|^2_{\mathbf{Q_f}}, \tag{4}$$

where $\|\cdot\|^2_{\mathbf{Q_f}} = (\cdot)^T\mathbf{Q_f}^{-1}(\cdot)$, and $\mathbf{Q_f}$ is the variance-covariance matrix of the double-difference carrier phase observables. Clearly, the more accurate estimations about $\mathbf{x}$ and $\mathbf{b}$, the higher precision positioning that GNSS can achieve.

B. The ILS Problem in Double-Difference Carrier-Phase Measurement

Given (4), parameter estimations of $\mathbf{x}$ and $\mathbf{b}$ are normally carried out in three steps — the float solution, the integer ambiguity estimation and the fixed solution respectively. Firstly, the float solution solves (4) but in an ordinary unconstrained least-squares formation with $\mathbf{x} \in \mathbb{Z}^n$ replaced by $\mathbf{x} \in \mathbb{R}^n$. As a result, both the real-valued estimates of $\mathbf{x}$ and $\mathbf{b}$, referred to as $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{b}}$, are obtained together with their corresponding variance-covariance matrices $\mathbf{Q}_{\widetilde{\mathbf{x}}}$ and $\mathbf{Q}_{\widetilde{\mathbf{b}}}$. Then, the integer ambiguity estimation at the second step tries to solve the integer least-squares (ILS) problem in the following

$$\mathbf{x}_{\text{optimal}} = \arg\min_{\mathbf{x}\in\mathbb{Z}^n} (\mathbf{x} - \widetilde{\mathbf{x}})^T\mathbf{Q}_{\widetilde{\mathbf{x}}}^{-1}(\mathbf{x} - \widetilde{\mathbf{x}}). \tag{5}$$

Finally, the baseline coordinates $\widetilde{\mathbf{b}}$ is adjusted accordingly by the integer ambiguity $\mathbf{x}_{\text{optimal}}$, thereby forming the solution to (4). Undoubtedly, the difficulty of solving the mixed least-squares problem in (4) principally lies on solving the integer least-square problem in (5), making it the key for the high precision positioning of GNSS.

Specifically, the ILS problem in (5) can be solved by a discrete search over an ellipsoidal space, where the shape and orientation of such an ellipsoid are determined by the variance covariance matrix of the integer variables. For this reason, the decorrelation in LAMBDA tries to seek out an equivalent ellipsoid but is much sphere-like, thus making the search within it more efficiently. More precisely, through the integer unimodular matrix $\mathbf{Z} \in \mathbb{Z}^{n\times n}$ with $|\det(\mathbf{Z})| = 1$, the ILS problem in (5) becomes

$$\widehat{\mathbf{z}} = \arg\min_{\mathbf{z}\in\mathbb{Z}^n} (\mathbf{z} - \widetilde{\mathbf{z}})^T\mathbf{Q}_{\widetilde{\mathbf{z}}}^{-1}(\mathbf{z} - \widetilde{\mathbf{z}}) \tag{6}$$

with $\mathbf{z} = \mathbf{Z}^T\mathbf{x}$, $\widetilde{\mathbf{z}} = \mathbf{Z}^T\widetilde{\mathbf{x}}$, $\mathbf{Q}_{\widetilde{\mathbf{z}}} = \mathbf{Z}^T\mathbf{Q}_{\widetilde{\mathbf{x}}}\mathbf{Z}$, and the correlation among components of $\mathbf{z}$ becomes less correlated than that of $\mathbf{x}$ through this transformation[2]. Then, based on (6), the searching stage of LAMBDA is performed to enumerate a subspace in $\mathbb{Z}^n$ by satisfying

$$(\mathbf{z} - \widetilde{\mathbf{z}})^T\mathbf{Q}_{\widetilde{\mathbf{z}}}^{-1}(\mathbf{z} - \widetilde{\mathbf{z}}) \leq \chi^2, \tag{7}$$

which contains the optimal solution $\mathbf{z}_{\text{optimal}}$ within it. Here $\chi > 0$ represents a constant to depict the sphere radius in the space.

[1] Normally, one of the receivers is the base station nearby while the other one is the target position no matter it is stationary or moving.

[2] There are many methods (e.g., LLL reduction) to realize this transformation by providing various solutions of $\mathbf{Z}$ for a more diagonalized variance-covariance matrix $\mathbf{Q}$ [38].

After $\mathbf{z}_{\text{optimal}}$ is determined, $\mathbf{x}_{\text{optimal}}$ in (5) can be got efficiently through $\mathbf{x}_{\text{optimal}} = \mathbf{Z}^{-T}\mathbf{z}_{\text{optimal}}$.

III. New Paradigm of the ILS Problem

Intuitively, given the ILS problem in (5), it is not hard to transfer it into an equivalent form as

$$\mathbf{x}_{\text{optimal}} = \arg\min_{\mathbf{x}\in\mathbb{Z}^n} \|\mathbf{c} - \mathbf{G}\mathbf{x}\|^2 \tag{8}$$

due to

$$\begin{aligned}\|\mathbf{c} - \mathbf{G}\mathbf{x}\|^2 =& (\mathbf{x} - \widetilde{\mathbf{x}})^T\mathbf{G}^T\mathbf{G}(\mathbf{x} - \widetilde{\mathbf{x}})\\ &+ \mathbf{c}^T(\mathbf{I} - \mathbf{G}(\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T)\mathbf{c}\end{aligned} \tag{9}$$

with $\widetilde{\mathbf{x}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{c}$ and $\mathbf{G}^T\mathbf{G} = \mathbf{Q}_{\widetilde{\mathbf{x}}}^{-1}$. This is also the methodology applied in [21], [25], [31], where the optimal solution $\mathbf{x}_{\text{optimal}}$ in (8) accounts for the results under the maximal likelihood (ML) criterion. Moreover, for a better illustration on the searching process, a QR-decomposition with $\mathbf{G} = \mathbf{Q}\mathbf{R}$ is applied and we express the system model in (8) as $\mathbf{y} = \mathbf{Q}^T\mathbf{c}$, where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix. Accordingly, this corresponds to

$$\mathbf{x}_{\text{optimal}} = \arg\min_{\mathbf{x}\in\mathbb{Z}^n} \|\mathbf{R}\mathbf{x} - \mathbf{y}\|^2. \tag{10}$$

A. Solving ILS Problem by Sphere Decoding

Clearly, given the ILS problem in (10), it can be addressed by enumerating the integer candidates within a sphere radius $\chi > 0$

$$\|\mathbf{R}\mathbf{x} - \mathbf{y}\| \leq \chi, \tag{11}$$

which is known as the *sphere decoding* (SD) in mathematics [30], [32].

Thanks to the special structure contributed by QR-decomposition, the discrete searching of sphere decoding based on (11) can be simply expressed as

$$|\widehat{x}_i - \widetilde{x}_i| \leq \left[\chi^2 - \sum_{j=i+1}^{n}\left|y_j - \sum_{l=j}^{n} r_{j,l}\widehat{x}_l\right|^2\right]^{\frac{1}{2}} / |r_{i,i}| \tag{12}$$

in a backwards order layer by layer (i.e., $i = n, n-1, \ldots, 1$), where

$$\widetilde{x}_i = \frac{y_i - \sum_{j=i+1}^{n} r_{i,j}\widehat{x}_j}{r_{i,i}}. \tag{13}$$

Here, at the i-th searching layer, the integer candidate $\widehat{x}_i^j$ who denotes the j-th closest integer candidate node to $\widetilde{x}_i$ will be saved if it satisfies (12). Finally, among the collected integer candidate vectors, the one with the closest Euclidean distance $\|\mathbf{R}\widehat{\mathbf{x}} - \mathbf{y}\|$ will be outputted as the solution. This is actually known as Fincke-Pohst strategy in sphere decoding [27]. At this point, we should mention that the above searching process is essentially the same with that in LAMBDA method, in which *Cholesky decomposition* is used to perform the related searching upon (5) layer by layer. Nevertheless, we emphasize that such a transformation of the ILS problem from (5) to (10) is meaningful as it paves the way for the analysis of the following work.

Intuitively, the selection of χ becomes a vital point in ILS estimator. In particularly, if χ is set too small, no candidate vectors can be found, and the searching has to restart again but with a larger size of χ. On the other hand, a large size χ would lead to considerable complexity waste. Although a few suggestions have been given for a reasonable choice of χ, solving a high-dimensional ILS problem within a limited time span still turns out to be impractical [33]. One inherent problem arising here is that there is only one parameter χ to coordinate the balance between performance and complexity, making such a trade-off untractable in theory. To this end, we propose to reformulate the paradigm of ILS problem, thus leading to more degrees of freedom in characterizing the trade-off between performance and complexity.

B. ILS Problem Transformation

The ILS problem in (5) or (10) can be viewed as an optimization problem, which attempts to return the target $\mathbf{x}$ with the smallest Euclidean distance $\|\mathbf{R}\mathbf{x} - \mathbf{y}\|$. To expand its capability for interpreting the ILS problem, we now introduce the following function

$$F(\mathbf{x}) = e^{-\frac{1}{2\sigma^2}\|\mathbf{R}\mathbf{x}-\mathbf{y}\|^2} \tag{14}$$

with $\sigma > 0$ to establish the new paradigm for the ILS problem transformation, i.e.,

$$\mathbf{x}_{\text{optimal}} = \arg\max_{\mathbf{x}\in\mathbb{Z}^n} F(\mathbf{x}) = \arg\min_{\mathbf{x}\in\mathbb{Z}^n} \|\mathbf{R}\mathbf{x} - \mathbf{y}\|^2, \tag{15}$$

which corresponds to finding the integer candidate vector $\mathbf{x}$ with the largest value in the function $F(\cdot)$.

According to such a transformation, the parameter standard deviation $\sigma > 0$ is introduced as the above paradigm equivalence always holds no matter what σ is, which means extra degrees of freedom can be obtained to interpret the system trade-off between performance and complexity. More precisely, a small σ is able to intensively enlarge the value difference among $F(\mathbf{x})$'s while a large σ will enable a more uniform values of $F(\mathbf{x})$'s. With respect to returning the optimal solution $\mathbf{x}_{\text{optimal}}$, it is strongly encouraged to use a small σ to emphasize the value difference between $F(\mathbf{x}_{\text{optimal}})$ and others. Note that $\mathbf{x}$ and $F(\mathbf{x})$ are one-to-one correspondence, based on the enhanced $F(\mathbf{x}_{\text{optimal}})$, one can enumerate over the function $F(\mathbf{x})$ through a value threshold

$$F(\mathbf{x}) = e^{-\frac{1}{2\sigma^2}\|\mathbf{R}\mathbf{x}-\mathbf{y}\|^2} \geq \frac{1}{K} \tag{16}$$

with initial searching size $K > 1$ to pick up the optimal solution $\mathbf{x}_{\text{optimal}}$.

In particular, (16) is similar to (11) but characterized by two parameters K and σ rather than the single one χ. More specifically, (16) can be rewritten as

$$\|\mathbf{R}\mathbf{x} - \mathbf{y}\| \leq \sigma\sqrt{2\ln K} \triangleq \chi, \tag{17}$$

which means the sphere radius χ in (11) is exactly interpreted by K and σ. In other words, the traditional Fincke-Pohst SD based on (12) can be applied directly for any given K and σ but it fails to fully take advantage of the two parameters K and σ to explore more. Therefore, with respect to the paradigm in (15), how to design the related searching algorithm to fully make use of K and σ becomes the key to exploit the system potential.

IV. Equivalent Searching Algorithm

In this section, inspired by the proposed paradigm of solving the ILS problem, we investigate how to utilize K and σ to perform the specific searching process. First of all, the equivalent searching algorithm (ESA) is proposed, which is essentially the same with Fincke-Pohst SD but parameterized by the initial searching size K and the standard deviation σ. Based on it, since extra degrees of freedom are obtained, the relationship between performance and complexity can be revealed, thus leading to a tractable trade-off in solving the ILS problem.

A. Algorithm Description

First of all, similar to Fincke-Pohst SD, the tree-search structure in a backwards order from $i = n$ to $i = 1$ layer by layer is retained. Then, based on it, the *searching size* $K(\widehat{x}_i^j)$ for each integer candidate node $\widehat{x}_i^j$ is defined as

$$K(\widehat{x}_i^j) \triangleq K(\underline{\widehat{x}_i^j}) \cdot f(\widehat{x}_i^j) \tag{18}$$

where the defined weighted function is

$$f(\widehat{x}_i^j) \triangleq e^{-\frac{1}{2\sigma_i^2}\|\widehat{x}_i^j - \widetilde{x}_i\|^2} \tag{19}$$

with $\sigma_i = \sigma/r_{i,i}$. Here, $\widehat{x}_i^j \in \mathbb{Z}$ with $j \in \{1, 2, \ldots\}$ denotes the j-th closest integer candidate node to $\widetilde{x}_i$, $\underline{\widehat{x}_i^j}$ represents the parent node of $\widehat{x}_i^j$ at the last searching layer $i+1$ and several children candidate nodes $\widehat{x}_i^j$ may share a same parent node $\underline{\widehat{x}_i^j}$. Moreover, the children candidate node $\widehat{x}_i^j$ will be retained only if the following *searching threshold*

$$K(\widehat{x}_i^j) \geq 1 \tag{20}$$

is satisfied. Otherwise, $\widehat{x}_i^j$ as a candidate node will be pruned, and the searching moves to the next layer based on those survived candidate nodes. As clearly shown in Fig. 2, candidate node $\widehat{x}_{i-1}^3$ is pruned due to $K(\widehat{x}_{i-1}^3) < 1$ while candidate nodes $\widehat{x}_{i-1}^1$ and $\widehat{x}_{i-1}^2$ are saved by satisfying the searching threshold.

Clearly, in this way, the initial searching size K actually corresponds to the root searching size for the following operations, i.e., $K = K(\underline{\widehat{x}_n^j})$. More specifically, the initial searching size K essentially serves as a parameter to adjust the searching threshold flexibly, where a larger size K amounts for a smaller searching threshold at each layer. Finally, the integer candidate vector with the closest Euclidean distance $\| \mathbf{R}\widehat{\mathbf{x}} - \mathbf{y} \|$ in the candidate list will be outputted as the estimator solution.

Theorem 1: The proposed equivalent searching algorithm is exactly the same with Fincke-Pohst SD but with sphere radius being characterized by K and σ

$$\chi_{\text{equivalent}} = \sigma\sqrt{2\ln K}. \tag{21}$$

Proof: From (18) and (20), the searching threshold can be further specified as

$$f(\widehat{x}_i^j) \geq \frac{1}{K(\underline{\widehat{x}_i^j})} = \frac{1}{K \cdot f(\widehat{x}_{i+1}^j) \cdots f(\widehat{x}_n^j)}. \tag{22}$$

According to (22), for any integer candidate vector $\widehat{\mathbf{x}}$ being collected by the proposed algorithm, the following relationship holds

$$f(\widehat{x}_1) \geq \frac{1}{K \cdot f(\widehat{x}_2) \cdots f(\widehat{x}_n)}, \tag{23}$$

which leads to

$$\begin{aligned} \prod_{i=1}^{n} f(\widehat{x}_{n-i+1}) &= \prod_{i=1}^{n} e^{-\frac{1}{2\sigma_{n-i+1}^2}\|\widehat{x}_{n-i+1} - \widetilde{x}_{n-i+1}\|^2} \\ &= e^{-\frac{1}{2\sigma^2}\|\mathbf{R}\mathbf{x} - \mathbf{y}\|^2} \\ &\geq \frac{1}{K}. \end{aligned} \tag{24}$$

Clearly, this corresponds to obtain the integer candidate vectors within the sphere radius

$$\|\mathbf{R}\mathbf{x} - \mathbf{y}\| \leq \sigma\sqrt{2\ln K}, \tag{25}$$

which is exactly the same to (17) in the new paradigm of ILS. ■

According to Theorem 1, the proposed equivalent searching algorithm is capable of collecting all the integer candidate vectors within the sphere radius $\chi_{\text{equivalent}} = \sigma\sqrt{2\ln K}$ by fully taking advantages of K and σ. Therefore, extra degrees of freedom are introduced to reexamine the ILS problem. Moreover, it is also straightforward to verify the equivalence of the proposed equivalent searching algorithm and Fincke-Pohst SD by showing they have the same searching space of $\widehat{x}_i$ at each layer. In particular, according to the searching threshold shown in (22), the searching space of $\widehat{x}_i$ given $\widetilde{x}_i$ can be derived as

$$|\widehat{x}_i - \widetilde{x}_i|_{\text{equivalent}} \leq \left[2\sigma^2 \ln K - \sum_{j=i+1}^{n} \left| y_j - \sum_{l=j}^{n} r_{j,l}\widehat{x}_l \right|^2\right]^{\frac{1}{2}} / |r_{i,i}|, \tag{26}$$

which is exactly the boundary of $|\widehat{x}_i - \widetilde{x}_i|_{\text{Fincke-Pohst}}$ in (12) for $1 \leq i \leq n$ by letting $\chi = \sigma\sqrt{2\ln K}$.

B. Trade-Off Analysis Between Performance and Complexity

Clearly, given the parameters K and σ, the proposed equivalent searching algorithm is able to obtain integer candidate vectors within a sphere radius $\chi_{\text{equivalent}} = \sigma\sqrt{2\ln K}$. Interestingly, note that in this searching process, either K or σ can be adjusted freely to control the size of sphere radius $\chi_{\text{equivalent}}$, which provides a more feasible way for the analytical diagnose in both performance and complexity of solving the ILS problem.

Lemma 1: In the proposed equivalent searching algorithm, for each parent candidate node $\underline{\widehat{x}_i^j}$ with $K(\underline{\widehat{x}_i^j}) \geq 1$, the number of its saved children candidate nodes at searching layer i, denoted by K_{save}, satisfies

$$K_{\text{save}} \leq K(\underline{\widehat{x}_i^j}) \tag{27}$$

for $\sigma \leq \min_i |r_{i,i}|/(2\sqrt{\pi})$.

Proof: Based on the searching threshold given in (20), the condition shown in (27) holds if the $\lfloor K(\underline{\widehat{x}_i^j})+1\rfloor$-th closest integer candidate to $\widetilde{x}_i$ will be pruned for sure, that is

$$K(\underline{\widehat{x}_i^j}) f(\widehat{x}_i^{\lfloor K(\underline{\widehat{x}_i^j})+1\rfloor}) < 1. \tag{28}$$

Next, because the distance $|\widehat{x}_i^j - \widetilde{x}_i|$ is bounded by

$$(j-1)\cdot\frac{1}{2} \leq |\widehat{x}_i^j - \widetilde{x}_i| \leq j\cdot\frac{1}{2}, \tag{29}$$

the condition in (28) can be guaranteed if

$$K(\underline{\widehat{x}_i^j})\cdot e^{-\frac{1}{8\sigma_i^2}(\lfloor K(\underline{\widehat{x}_i^j})+1\rfloor-1)^2} < 1, \tag{30}$$

or equivalently

$$\sigma^2 < \frac{(\lfloor K(\underline{\widehat{x}_i^j})+1\rfloor-1)^2}{8\ln K(\underline{\widehat{x}_i^j})}\cdot r_{i,i}^2. \tag{31}$$

Subsequently, it is straightforward to justify that the right-hand term of (31) is lower bounded as

$$\frac{(\lfloor K(\underline{\widehat{x}_i^j})+1\rfloor-1)^2}{8\ln K(\underline{\widehat{x}_i^j})}\cdot r_{i,i}^2 > \frac{1}{8\ln 2}\cdot r_{i,i}^2. \tag{32}$$

Therefore, by setting $\sigma \leq \min_i |r_{i,i}|/(2\sqrt{\pi})$ for $1 \leq i \leq n$, it it easy to check the condition in (31) is satisfied, which means the $\lfloor K(\underline{\widehat{x}_i^j})+1\rfloor$-th closest integer candidate to $\widetilde{x}_i$ will definitely be pruned so that $K_{\text{save}} \leq K(\underline{\widehat{x}_i^j})$. ■

Lemma 2: In the equivalent searching algorithm, for each parent candidate node $\underline{\widehat{x}_i^j}$ with $K(\underline{\widehat{x}_i^j}) \geq 1$, the summation of searching sizes of its children candidate nodes at searching layer i is non-increasing, i.e.,

$$\sum_j K(\widehat{x}_i^j) < K(\underline{\widehat{x}_i^j}) \tag{33}$$

for $\sigma \leq \min_i |r_{i,i}|/(2\sqrt{\pi})$.

Proof: According to the definition of $K(\widehat{x}_i^j)$ in (18), the summation of searching sizes of its children candidate nodes at searching layer i follows that

$$\begin{aligned}\sum_j K(\widehat{x}_i^j) &= K(\underline{\widehat{x}_i^j})\cdot\sum_j f(\widehat{x}_i^j)\\ &< K(\underline{\widehat{x}_i^j})\cdot\sum_{\widehat{x}_i\in\mathbb{Z}} e^{-\frac{1}{2\sigma_i^2}\|\widehat{x}_i-\widetilde{x}_i\|^2}\\ &\overset{(a)}{\leq} K(\underline{\widehat{x}_i^j})\cdot\sum_{\widehat{x}_i\in\mathbb{Z}} e^{-\frac{1}{2\sigma_i^2}\|\widehat{x}_i\|^2}\\ &\overset{(b)}{=} K(\underline{\widehat{x}_i^j})\cdot\vartheta_3(|r_{i,i}|^2/2\pi\sigma^2)\\ &\overset{(c)}{\approx} K(\underline{\widehat{x}_i^j}).\end{aligned} \tag{34}$$

Here, inequality (a) holds due to the following relationship

$$\sum_{\widehat{x}_i\in\mathbb{Z}} e^{-\frac{1}{2\sigma_i^2}\|\widehat{x}_i-\widetilde{x}_i\|^2} \leq \sum_{\widehat{x}_i\in\mathbb{Z}} e^{-\frac{1}{2\sigma_i^2}\|\widehat{x}_i\|^2} \tag{35}$$

where the equality happens only when $\widetilde{x}_i \in \mathbb{Z}$. Inequality (b) recalls the *Jacobi theta function* ϑ_3 [39]

$$\vartheta_3(\nu) = \sum_{n=-\infty}^{+\infty} e^{-\pi\nu n^2} \tag{36}$$

while the approximation in (c) obeys

$$\vartheta_3(|r_{i,i}|^2/2\pi\sigma^2) \leq \vartheta_3(2) = 1.0039 \approx 1 \tag{37}$$

for $\sigma \leq \min_i |r_{i,i}|/(2\sqrt{\pi})$ as $\vartheta_3(\nu)$ is monotonically decreasing with $\nu > 0$. ■

Based on Lemmas 1 & 2, the complexity of the proposed equivalent searching algorithm can be derived by means of the number of visited nodes as follows.

Theorem 2: In the equivalent searching algorithm, let $\sigma \leq \min_i |r_{i,i}|/(2\sqrt{\pi})$, the number of visited nodes denoted by $|S|$ is upper bounded by

$$|S| < nK. \tag{38}$$

Proof: First of all, based on Lemma 1, the number of saved candidate nodes at searching layer $1 \leq i \leq n$, i.e., $K_{\text{save}}^{\text{layer } i}$, is upper bounded by the summation of searching sizes at the previous layer $i+1$,

$$K_{\text{save}}^{\text{layer } i} = \sum K_{\text{save}} \leq \sum K(\underline{\widehat{x}_i^j}) = K_{\text{searching size}}^{\text{layer } i+1}. \tag{39}$$

On the other hand, according to Lemma 2, it is easy to check that the summation of searching sizes at each searching layer is decreasing as

$$K_{\text{searching size}}^{\text{layer } 1} < \ldots < K_{\text{searching size}}^{\text{layer } n} < K_{\text{searching size}}^{\text{layer } n+1} = K \tag{40}$$

so that the number of visited nodes in the proposed equivalent searching algorithm is upper bounded by

$$|S| = \sum_{i=1}^{n} K_{\text{save}}^{\text{layer } i} < n\cdot K_{\text{searching size}}^{\text{layer } n+1} = nK, \tag{41}$$

completing the proof. ■

The clear complexity upper bound for the proposed equivalent searching algorithm in Theorem 2 answers a fundamental question in solving the ILS problem since it is rather difficult to estimate the required complexity based on the single parameter χ. Instead, based on K and σ, the trade-off between performance and complexity turns out to be analytical. Therefore, one can simply fix $\sigma = \min_i |r_{i,i}|/(2\sqrt{\pi})$ and enjoy the trade-off by tuning the initial searching size $K > 1$, which leads us to the following Theorem.

Theorem 3: In the equivalent searching algorithm, let $\sigma = \min_i |r_{i,i}|/(2\sqrt{\pi})$, the performance measured by the sphere radius $\chi_{\text{equivalent}} = \sqrt{\frac{\ln K}{2\pi}}\min_i |r_{i,i}|$ corresponds to the tractable complexity upper bounded by $|S| < nK$.

According to Theorem 3, in order to achieve a larger sphere radius $\chi_{\text{equivalent}}$, the initial searching size K should increase accordingly, which also implies more visited nodes being upper bounded by nK. Subsequently, consider solving the ILS problem in (10), the required initial searching size K as well as the complexity $|S|$ can be derived in the following.

Theorem 4: To solve the ILS problem, the initial searching size K by the proposed equivalent searching algorithm is $e^{2\pi\|\mathbf{R}\mathbf{x}_{\text{optimal}}-\mathbf{y}\|^2/\min_i^2|r_{i,i}|}$, which corresponds to the complexity upper bounded by $|S| < n \cdot e^{2\pi\|\mathbf{R}\mathbf{x}_{\text{optimal}}-\mathbf{y}\|^2/\min_i^2|r_{i,i}|}$.

On the other hand, as the number of saved candidate nodes at the final searching layer $i = 1$ accounts for the number of collected integer candidate vectors, i.e., $K_{\text{save}}^{\text{layer }1} = |L|$, the following result can be achieved.

Corollary 1: The number of integer candidate vectors collected by the proposed equivalent searching algorithm, denoted by $|L|$, is upper bounded by

$$|L| < K. \tag{42}$$

Corollary 1 is rather meaningful for the study of ILS by bounding the number of collected integer candidate vectors. To the best of our knowledge, this is the first time that the number of visited integer candidate vectors can be quantized, thus making the related ILS estimator more tractable and analytical. Here, we also point out the significance of LLL reduction, which is able to effectively improve the value of $\min_i |r_{i,i}|$ (i.e., $\min_i \|\widehat{\mathbf{b}}_i\|$, where $\widehat{\mathbf{b}}_i$'s are the Gram-Schmidt vectors of the lattice basis $\mathbf{B}$) through the matrix transformation [19], [40].

$$\mathbf{z}_{\text{optimal}} = \arg\min_{\mathbf{x}\in\mathbb{Z}^n} \|\overline{\mathbf{R}}\mathbf{z} - \mathbf{y}\|^2, \tag{43}$$

where $\mathbf{x}$ and $\mathbf{z}$ are one-to-one correspondence according to $\overline{\mathbf{R}} = \mathbf{R}\mathbf{U}, \mathbf{z} = \mathbf{U}^{-1}\mathbf{x}, \mathbf{U} \in \mathbb{Z}^n$, and $|\mathbf{U}| = 1$ is the unimodular matrix produced by LLL reduction.

V. Further Enhancements of the Equivalent Searching Algorithm

Since the summation of searching sizes is decreasing layer by layer, there is a potential risk in the equivalent searching algorithm: it only works well when the initial searching size K is large enough. As for small value K, the searching still performs but it may terminate at early layers because all the possible candidate nodes are pruned due to a limited searching size $K(\widehat{x}_i^j)$. Actually, such a problem does also exist in Fincke-Pohst SD, where no valid solutions will be outputted by a small sphere radius χ. In this condition, even though considerable complexity cost has been consumed for searching in the early layers, no integer candidate vectors can be saved at the end, making the searching process meaningless. This in essence poses an inherent question to the proposed equivalent searching algorithm: how to deterministically achieve the performance by yielding valid solutions when K is small? In what follows, we try to answer this question by rescuing a few valuable integer candidate vectors from the mechanism of searching threshold.

A. Candidate Protection

In order to maintain a desired performance of the proposed equivalent searching algorithm for small sizes of K, the *candidate protection* mechanism is proposed. Specifically, for a candidate node $\widehat{x}_i^j$ with small searching size

$$1 \leq K(\widehat{x}_i^j) < e^{\frac{\pi}{2\min^2|r_{i,i}|}}, \tag{44}$$

the searching along this branch in the following layers is broke off, while the closest candidate nodes $\widehat{x}_{i-1}^1, \ldots, \widehat{x}_1^1$ in the rest of layers are computed directly, thereby outputting an integer candidate vector $\widehat{\mathbf{x}}$. The motivation behind this mechanism is mainly because the searching solution consists of the closest candidate nodes $\widehat{x}_{i's}^1$ in the rest of layers normally has the largest sampling probability along that branch. Therefore, rather than removing those candidate nodes with small searching sizes directly, the candidate protection mechanism effectively extends the searching threshold in (20) by reserving some high-qualified integer candidate vectors.

Remark 1: With $\sigma = \min_i |r_{i,i}|/(2\sqrt{\pi})$, for parent candidate node $\widehat{x}_i^j$ with $K(\widehat{x}_i^j) \geq e^{\frac{\pi}{2\min^2|r_{i,i}|}}$, there does exist its children node according to the pruning threshold. On the contrary, the children node existence of the parent candidate node $\widehat{x}_i^j$ with $1 \leq K(\widehat{x}_i^j) < e^{\frac{\pi}{2\min^2|r_{i,i}|}}$ is not guaranteed so that the candidate protection is invoked.

Note that the added candidate protection works compatibly with the searching threshold as it tries to rescue a few candidate vectors discarded by the latter. Clearly, the usage of candidate protection expands the initial searching size $K > 1$ to $K \geq 1$. Interestingly, when $K = 1$, the performance of the *Babai's nearest plane* algorithm will be achieved by the proposed algorithm, which establishes a flexible performance tradeoff through tuning K. Therefore, candidate protection can be simply carried out through Babai's nearest plane algorithm since $\widehat{x}_1^1, \ldots, \widehat{x}_n^1$ is exactly the decoding result of it. By doing this, the final candidate list L consists of two parts: integer candidate vectors collected by candidate protection, i.e., $L_{\text{protection}}$ and searching threshold, i.e., $L_{\text{searching}}$, where the candidate vector with the closest Euclidean distance $\| \mathbf{R}\widehat{\mathbf{x}} - \mathbf{y} \|$ among the set $L = L_{\text{protection}} \bigcup L_{\text{searching}}$ is outputted as the final solution. Intuitively, even though $L_{\text{searching}} = \emptyset$, there are still searching outputs due to $L_{\text{protection}}$.

The update by the proposed candidate protection is meaningful as it converts the ESA from an optimal scheme (i.e., when K is sufficient) to a suboptimal scheme (i.e., when K is limited). It yields the suboptimal solutions rather than restart the searching with a larger K, which not only fully exploits the complexity cost but also achieves a better performance than the traditional LAMBDA algorithm. As shown in Fig. 3, according to the candidate protection, the candidate node $\widehat{x}_{i-1}^2$ with $1 \leq K(\widehat{x}_i^2) < e^{\frac{\pi}{2\min^2|r_{i,i}|}}$ is saved to yield the candidate vector $\widehat{\mathbf{x}}_1$ directly while the candidate node $\widehat{x}_{i-1}^3$ is pruned due to $K(\widehat{x}_i^3) < 1$.

To summarize, at each searching layer, the proposed equivalent searching algorithm operates in the following two steps

- *Compute the searching size $K(\widehat{x}_i^j)$ by (18);*
- *Obtain candidate nodes $\widehat{x}_i^j$ by (20). If $1 \leq K(\widehat{x}_i^j) < e^{\frac{\pi}{2\min^2|r_{i,i}|}}$, invoke the Babai's nearest plane algorithm to directly return a searching candidate $\widehat{\mathbf{x}}$,*

Algorithm 1: Babai's Nearest Plane Algorithm.

Require: $\mathbf{G} \in \mathbb{R}^{n\times n}, \mathbf{c} \in \mathbb{R}^n$
Ensure: $\mathbf{x} \in \mathbb{Z}^n$
let $\mathbf{G} = \mathbf{QR}$ and $\mathbf{y} = \mathbf{Q}^\dagger \mathbf{c}$
for $i = n, ..., 1$ **do**
  let $\widetilde{x}_i = \frac{y_i - \sum_{j=i+1}^n r_{i,j}x_j}{r_{i,i}}$
  get $\widehat{x}_i$ by directly rounding as $\widehat{x}_i = \lceil \widetilde{x}_i \rfloor$
end for
return $\mathbf{x}$

where the implementation of the Babai's nearest plane algorithm for candidate protection is outlined in Algorithm 1.

B. Complexity Analysis

Next, we show that even with the candidate protection, the complexity by means of the number of visited nodes $|S|$ and the number of collected candidates vectors $|L|$ still remain the same upper bound as before.

Theorem 5: Given the initial searching size $K \geq 1$ with $\sigma = \min_i |r_{i,i}|/(2\sqrt{\pi})$, the number of candidate vectors collected by the equivalent searching algorithm under both the searching threshold and the candidate protection is upper bounded by

$$|L| < K \tag{45}$$

with bounded total number of visited nodes

$$|S| < nK. \tag{46}$$

Proof: Because the collected integer candidate vectors come from *searching threshold* and *candidate protection* respectively, and because the summation of the searching size at each layer is decreasing, it follows that

$$\begin{aligned} K &= K(\underline{x_n^j}) \\ &> \sum K(x_n^{\text{protection}}) + \sum K(x_n^{\text{searching}}) \\ &> \sum K(x_n^{\text{protection}}) + \sum K(x_{n-1}^{\text{protection}}) + \sum K(x_{n-1}^{\text{searching}}) \\ &> \cdots \\ &> \sum_{i=2}^{n} \left[\sum K(x_i^{\text{protection}})\right] + \sum K(x_2^{\text{searching}}), \end{aligned} \tag{47}$$

where $1 \leq K(x_i^{\text{protection}}) < e^{\frac{\pi}{2\min^2 |r_{i,i}|}}$ and $K(x_2^{\text{searching}}) \geq e^{\frac{\pi}{2\min^2 |r_{i,i}|}}$.

According to candidate protection, only one integer candidate vector can be saved for each $K(x_i^{\text{protection}}), 2 \leq i \leq n$. Therefore, the number of integer candidate vectors obtained by candidate protection from searching layer n to 2 in all is upper bounded by

$$|L_{\text{protection}}| \leq \sum_{i=2}^{n} \left[\sum K(x_i^{\text{protection}})\right]. \tag{48}$$

Algorithm 2: Equivalent Searching Algorithm.

Require: $K, \mathbf{R}, \mathbf{y}, \sigma = \min_i |r_{i,i}|/(2\sqrt{\pi}), L = \emptyset$
Ensure: $\mathbf{x} \in \mathbb{Z}^n$
invoke **Function 1** with $i = n$ to search layer by layer
add all the candidates $\widehat{\mathbf{x}}$'s generated by **Function 1** to L
output $\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}\in L} \|\mathbf{y} - \mathbf{Rx}\|$ as the solution

Function 1: Searching at Layer i Given $[\widehat{x}_n, \ldots, \widehat{x}_{i+1}]$

compute $\widetilde{x}_i$ according to (13)
compute the probability $f(\widehat{x}_i^j)$ by (19) with $j \in [1, 2, 3]$
compute the searching size $K(\widehat{x}_i^j)$ according to (18)
for each specific integer candidate $\widehat{x}_i^j$ **do**
  if $K(\widehat{x}_i^j) < 1$ **then**
    prune $\widehat{x}_i^j$ from the tree search
  else
    save $\widehat{x}_i^j$ to form the searching result $[\widehat{x}_n, \ldots, \widehat{x}_{i+1}, \widehat{x}_i^j]$
    if $1 \leq K(\widehat{x}_i^j) < e^{\frac{\pi}{2\min^2 |r_{i,i}|}}$ **then**
      output candidate $\widehat{\mathbf{x}}$ by Babai's nearest plane algorithm
    else if $K(\widehat{x}_i^j) \geq e^{\frac{\pi}{2\min^2 |r_{i,i}|}}$ **then**
      if $i = 1$ **then**
        output the candidate $\widehat{\mathbf{x}}$
      else
        invoke **Function 1** to search the next layer $i - 1$
      end if
    end if
  end if
end for

Meanwhile, the number of candidate vectors collected by the searching threshold corresponds to $K_{\text{save}}^{\text{layer 1}}$ can be bounded as

$$|L_{\text{searching}}| = K_{\text{save}}^{\text{layer 1}} \leq \sum K(x_2^{\text{searching}}) \tag{49}$$

due to (39) in Theorem 2. Therefore, by simply substituting (48) and (49) into (47), we have

$$|L| = |L_{\text{searching}}| + |L_{\text{protection}}| < K. \tag{50}$$

Consequently, based on (50), the number of visited candidate nodes can be easily derived by

$$|S| < nK, \tag{51}$$

since all the visited nodes are counted to generate $|L|$ integer candidate vectors. ■

Therefore, it is clear to see that the proposed ESA establishes a flexible and tractable trade-off, which bridges the suboptimal and the optimal performance from complexity $|S| = n$ to $|S| < n \cdot e^{2\pi\|\mathbf{Rx}_{\text{optimal}} - \mathbf{y}\|^2/\min_i^2 |r_{i,i}|}$ by simply tuning K. Furthermore, according to Theorem 5, the computational complexity of the equivalent searching algorithm actually can be calibrated by the Babai's nearest plane algorithm (i.e., $O(n^2)$ with n visited candidate nodes). More specifically, it can be easily derived to be upper bounded by $O(K \cdot n^2)$ due to less than nK visited nodes. Therefore, to achieve a better performance, the computational complexity increases accordingly for a larger sphere radius $\chi_{\text{equivalent}} = \sqrt{\frac{\ln K}{2\pi}} \min_i |r_{i,i}|$. Nevertheless, once the optimal solution is ensured to return for a certain sphere radius, increasing K would turn out to be meaningless as the system performance would never improve anymore.

C. Complexity Reduction in Implementation

According to (18), the searching size $K(\widehat{x}_i^j)$ of a candidate integer node $\widehat{x}_i^j \in \mathbb{Z}$ decreases with the increment of $j \in \{1, 2, \ldots\}$. Thanks to this monotonicity, we now study the size of index j in the equivalent searching algorithm for practical implementation, which reduces the state space of j from $\{1, 2, \ldots\}$ to $\{1, \ldots, N\}$.

In particular, according to (18), the searching size $K(\widehat{x}_i^j)$ of a parent node is allocated to its children nodes by the weight $f(\widehat{x}_i^j)$. More specifically, let us rewrite $f(\widehat{x}_i^j)$ at searching layer i as

$$f(x_i^j) = \begin{cases} e^{-\frac{1}{2\sigma_i^2}((j-1)/2+d)^2} & \text{when } j \text{ is odd}, \\ e^{-\frac{1}{2\sigma_i^2}(\frac{j}{2}-d)^2} & \text{when } j \text{ is even}, \end{cases} \tag{52}$$

where $\frac{1}{2} \geq d = |x_i^1 - \widetilde{x}_i| \geq 0$. In this way, the summation of $f(\widehat{x}_i^j)$ for the first $2N$ candidate nodes with respect to $\widetilde{x}_i$ (i.e., $\widehat{x}_i^1, \ldots, \widehat{x}_i^{2N} \in \mathbb{Z}$) can be derived as

$$\sum_{j=1}^{2N} f(x_i^j) = \sum_{j=1}^{N} \left(e^{-\frac{1}{2\sigma_i^2}(j-1+d)^2} + e^{-\frac{1}{2\sigma_i^2}(j-d)^2} \right). \tag{53}$$

On the other hand, the weight except those $2N$ candidate nodes can be derived as

$$\begin{aligned} \sum_{j=2N+1}^{\infty} f(x_i^j) &= \sum_{j \geq N+1} \left(e^{-\frac{1}{2\sigma_i^2}(j-1+d)^2} + e^{-\frac{1}{2\sigma_i^2}(j-d)^2} \right) \\ &< \sum_{j \geq N+1} 2 \cdot e^{-\frac{1}{2\sigma_i^2}(j-1)^2} \\ &< \sum_{j \geq N+1} 2 \cdot e^{-\frac{1}{2\sigma_i^2}[(j-1)^2 - \frac{1}{4}]} \\ &= O\left(e^{-2\pi N^2}\right). \end{aligned} \tag{54}$$

Clearly, the tail bound in (54) decays exponentially fast due to $e^{2\pi} \gg 1$. Meanwhile, with $\sigma = \min_i |r_{i,i}|/(2\sqrt{\pi})$, from (37) it is easy to check

$$1 < \sum_{j=1}^{\infty} f(x_i^j) \leq \vartheta_3(2) = 1.0039, \tag{55}$$

so that the ratio

$$\sum_{j=2N+1}^{\infty} f(x_i^j) / \sum_{j=1}^{\infty} f(x_i^j) \tag{56}$$

becomes negligible with the increment of N.

Therefore, $j = 3$ is recommended in practice because the computation of $f(x_i^j)$ for $j > 3$ is valueless unless the initial searching size K is sufficiently large. To summarize, Overall, the proposed equivalent searching algorithm is presented in Algorithm 2, where enhancement mechanisms of candidate protection and complexity reduction are included for a better system trade-off.

VI. Numerical Simulations

In this section, to compare different search strategies for GNSS high-dimensional ambiguity resolution, the performance and complexity of the proposed equivalent searching algorithm (ESA) is evaluated by simulations based on MATLAB. Given the system model in (3), we consider its equivalent ILS version in (8). Therefore, the following system model is established

$$\mathbf{c} = \mathbf{G}\mathbf{x} + \mathbf{w}, \tag{57}$$

which is essentially the same with (5) via $\widetilde{\mathbf{x}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{c}$ and $\mathbf{G}^T\mathbf{G} = \mathbf{Q}_{\widetilde{\mathbf{x}}}^{-1}$. Here, we set $\mathbf{x} \in \mathcal{X}^n \subseteq \mathbb{Z}^n$, $\mathcal{X} = [-4, -3, -2, -1, 0, 1, 2, 3]$, and assume the matrix $\mathbf{G} \in \mathbb{R}^{n \times n}$ contains Gaussian-distributed components with unit variance. Meanwhile, $\mathbf{w}$ denotes the noise vector with zero mean and variance σ_w^2, so that the hardness of solving the ILS problem can be adjusted by tuning the value σ_w^2. Furthermore, to verify the system performance for solving (8) (i.e., (5)), the integer error probability (IEP) is applied to describe the percentage of error estimated integers ($\widehat{x}_i \neq x_i$) by Monte Carlo methods (i.e., computed by 2000 random variates),

$$P_{\text{IEP}} = \frac{\text{number of errors in recovering } \mathbf{x}}{\text{total number of } \mathbf{x} \text{ being recovered}} \tag{58}$$

where a smaller IEP corresponds to a better performance of the underlying ILS solver. Intuitively, if $\mathbf{x}_{\text{optimal}}$ could be returned, then the problem of integer phase ambiguity estimation in (4) for GNSS can be successfully solved. The smaller P_{IEP}, the more accurate GNSS positioning.[3] In this way, the performance difference between optimal and suboptimal schemes can be clearly illustrated in terms of IEP. As a flexible and tractable ILS solver to bridge the suboptimal and optimal performance, the proposed ESA enjoys the trade-off between performance and complexity through the initial searching size K.

Fig. 4 shows the integer error probabilities of the proposed equivalent searching algorithm (ESA) in 16×16 and 20×20 ILS systems with $\sigma_w^2 = 0.705$ and $\sigma_w^2 = 0.8812$ respectively. Here, the suboptimal LLL-reduction-aided Babai's nearest plane algorithm serves as a performance baseline and the LAMBDA algorithm from [14] based on the maximal likelihood (ML) criterion is added as the optimal performance. Note that Babai's

[3] Other metrics like success rate and voronoi region for performance evaluation also exist, see [41] and [34] for more details.

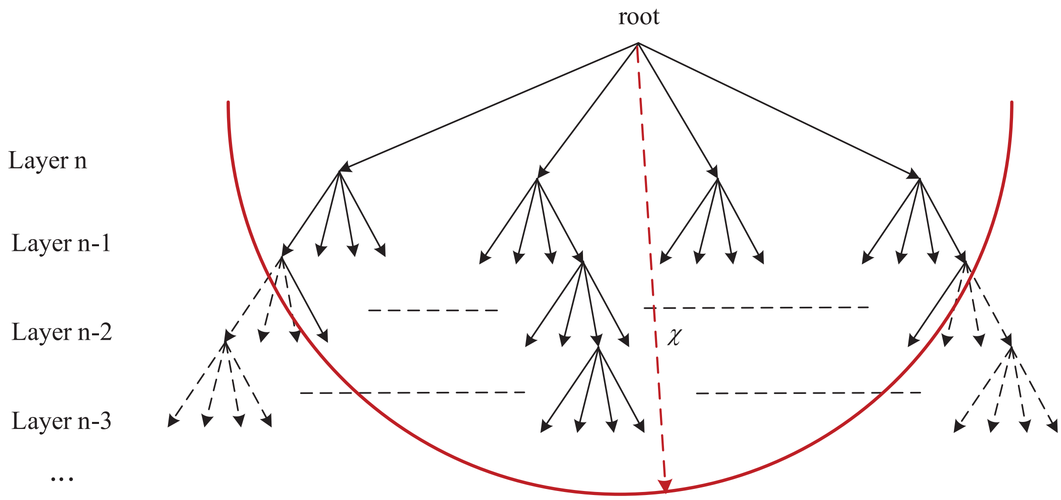

Fig. 1. Illustration of the sphere decoding for solving the ILS problem. The searching stage is carried out in a tree structure under the constraint of sphere radius χ.

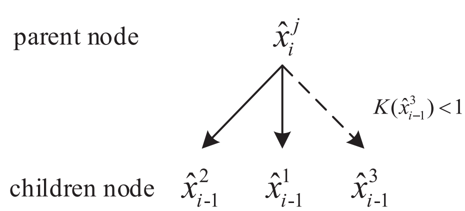

Fig. 2. Illustration of searching process via the searching threshold.

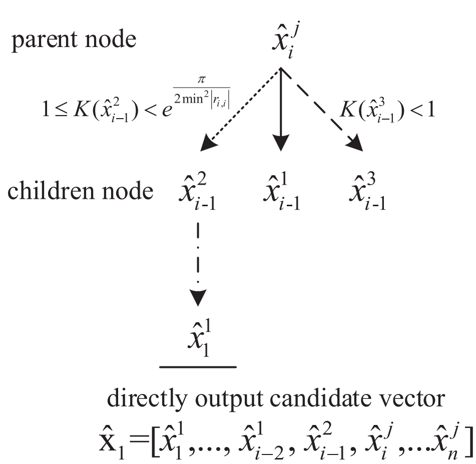

Fig. 3. Illustration of the candidate protection.

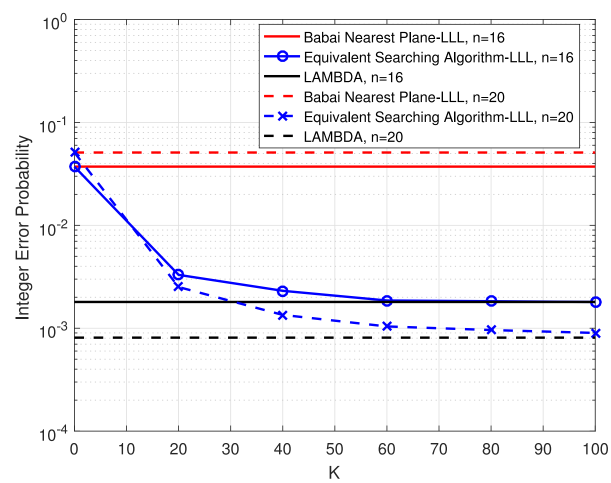

Fig. 4. Integer error probability versus initial searching size K for 16×16 and 20×20 ILS systems.

nearest plane algorithm actually works the same with the integer bootstrapping, where sequential processing is applied by full utilizing the correlations over elements of $\mathbf{x}$. As for the performance comparison in the 16×16 system, there is a substantial performance gap between suboptimal and optimal solutions for solving the ILS problem in (57). As expected, with the increment of K, the integer error probability of ESA decreases gradually for a better performance. Observe that with $K = 50$, the performance of ESA suffers negligible loss compared with the optimal solution. Therefore, with a moderate K, near-ML performance can be achieved. Of course, the system performance can be further improved by increasing K until the optimal solution is obtained by ESA. However, as demonstrated in Theorem 4, exponential increment of K would be required, which is not necessary in practice. On the other hand, with respect to cases of 20×20, due to the larger system dimension, a larger size K is needed to achieve the near-ML performance compared to curves in cases of 16×16. By adjusting the initial searching size K, the proposed equivalent searching algorithm enjoys a flexible trade-off between performance and complexity. This is rather meaningful in practice since the optimal solution can be well approximated by a moderate size K, especially for the application of real-time accurate positioning. Most importantly, the complexity of ESA can be tracked according to K.

For a better understanding, the comparison about the average number of the visited nodes $|S|$ obtained by equivalent searching algorithm for both 16×16 and 20×20 ILS systems is given in Fig. 5. In particular, the number of visited nodes invoked by the

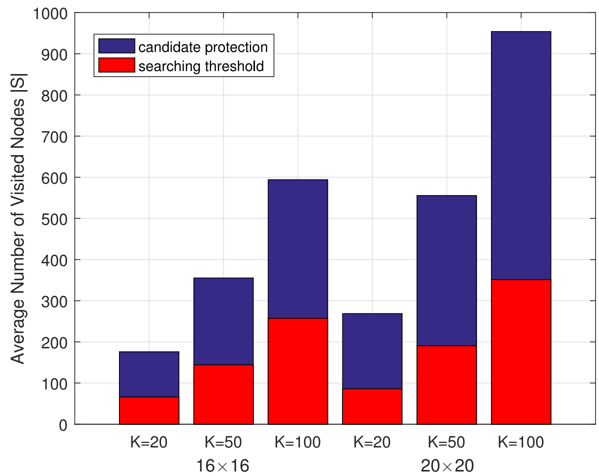

Fig. 5. Average number of visited nodes $|S|$ versus initial searching size K in 16×16 and 20×20 ILS systems.

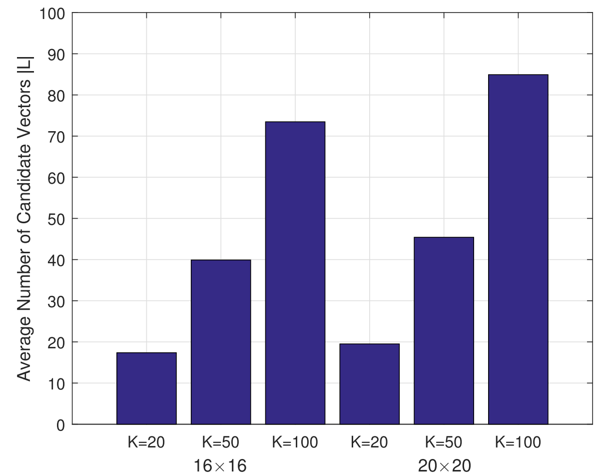

Fig. 6. Average number of collected candidate vectors $|L|$ versus initial searching size K in 16×16 and 20×20 ILS systems.

searching threshold, i.e., $|S|_{\text{searching}}$, is indicated by the red bar while the number of visited nodes invoked by the candidate protection, i.e., $|S|_{\text{protection}}$, is represented by the blue bar, thus forming the total number $|S| = |S|_{\text{searching}} + |S|_{\text{protection}}$ together. Intuitively, in both cases, we can see that $|S|_{\text{protection}} > |S|_{\text{searching}}$, implying the mechanism of candidate protection plays a more important role in outputting the candidate vectors. Meanwhile, this also means that most of searching attempts are terminated at the early stages. If the mechanism of candidate protection is removed, then few candidate vectors could be obtained due to the limited sphere radius, which is in line with the operation of the traditional sphere decoding. On the other hand, it is clear that $|S|$ increases with the initial searching size K while the increment of system dimensions n would also improve the number of visited nodes. This is straightforward to understand as the increase of n will greatly expand the tree structure in the searching process. Meanwhile, with the increase of K, the ratio $|S|_{\text{searching}}/|S|_{\text{protection}}$ improves accordingly, indicating a better searching capability of the proposed ESA algorithm. In addition, it is easy to verify that the total number of visited nodes $|S|$ follows the upper bound given in (46), thus providing a tractable complexity measurement for the proposed ESA algorithm.

On the other hand, Fig. 6 shows the complexity comparison of the proposed ESA in terms of the average number of the collected candidate vectors $|L|$ in both 16×16 and 20×20 ILS systems. In particular, as K increases, $|L|$ improves gradually as more qualified candidate vectors are obtained by the relaxed searching threshold. Clearly, $|L|$ is always much smaller than the initial searching size K, which is accordance with Theorem 5. More specifically, this means the upper bound $|L| < K$ is actually rather loose, where further modifications can be given to exploit the potential therein. Meanwhile, as expected, with the increase of the system dimension, more candidate vectors will be returned for the final output due to the larger depth of the tree structure. In addition, note that few eligible candidate vectors could be collected in most cases if the mechanism of candidate protection is removed, which is just the case of the traditional sphere decoding with limited sphere radius.

TABLE I
AVERAGE RUNNING FLOPS FOR COMPLEXITY COMPARISON IN ILS SYSTEMS WITH VARIOUS DIMENSIONS

	$n = 12$	$n = 16$	$n = 20$	$n = 24$
Babai-LLL	1947	4474	8581	14650
ESA (K=15)	3258	7762	15505	26210
ESA (K=50)	3529	8595	17260	29286
ESA (K=100)	3754	9312	18843	32462
LAMBDA	14352	94503	505692	2016523

Table I shows the complexity comparison in flops of the proposed equivalent searching algorithm with different system dimensions, where the flops evaluation scenario that we use comes from [42]. Clearly, the proposed ESA needs much lower flops than the optimal LAMBDA solution, and the complexity increases accordingly with the size of K. Note that the complexity also increases with the increment of system dimension but it can be well controlled by adjusting K. This is quite different from the optimal LAMBDA solution as its complexity increases exponentially with the system dimension, making it impractical especially for high-dimensional GNSS systems.

Following the same scenario in Table I, as a complement to illustrate the computational cost, Table II is given to show the complexity comparison in average elapsed running times for 100 runs. Typically, the simulation is conducted by MATLAB R2016a on a single computer, with an Intel Core i7 processor at 2.7 GHz, a RAM of 8 GB and Windows 10 Enterprise Service Pack operating system. As can be seen clearly, the average elapsed running times of the equivalent searching algorithm grow mildly with the increase of system dimension. On the

TABLE II
AVERAGE RUNNING TIMES FOR COMPLEXITY COMPARISON IN ILS SYSTEMS WITH VARIOUS DIMENSIONS

	$n = 12$	$n = 16$	$n = 20$	$n = 24$
Babai-LLL	0.582	0.661	0.738	0.797
ESA (K=15)	0.872	1.1354	1.465	1.902
ESA (K=50)	0.897	1.19	1.668	2.157
ESA (K=100)	0.935	1.298	1.826	2.325
LAMBDA	1.103	2.92	8.78	25

contrary, the optimal LAMBDA solution takes an exponentially increasing average elapsed running time, which is unaffordable in most of cases.

VII. CONCLUSION

In this paper, a new paradigm of solving the ILS problem has been adopted to the integer phase ambiguity estimation for high precision GNSS positioning of IIoT, where the original ILS problem originated from the integer phase ambiguity estimation is converted into an equivalent one but characterized by two parameters rather than one. Thanks to the introduced extra degrees of freedom in interpreting the system trade-off between performance and complexity, significant potential can be further exploited. Based on it, the equivalent searching algorithm has been proposed, which addresses the ILS problem tractably with a clear complexity upper bound. Furthermore, the equivalent searching algorithm has been upgraded by adding the candidate protection mechanism, which generalizes the searching results from optimal to sub-optimal solutions. In this way, the proposed equivalent searching algorithm suits well for various positioning requirements, where the positioning trade-off for GNSS is adjusted through the single parameter K freely.

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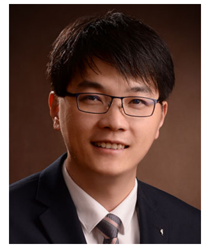

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